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DETERMINATION OF THE NATURAL FREQUENCIES OF AN ELLIPTIC SHELL OF CONSTANT THICKNESS BY THE FINITE-ELEMENT METHOD

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We study natural vibrations of a thin isotropic elliptic shell of constant thickness by the finite-element method realized with the use of the high-performance FEMAR software developed for the engineering analyses. The comparative analysis of the numerical results and experimental data is carried out.

Statement of the Problem

Noncircular cylindrical shells are extensively used in various fields of engineering. For the safe operation of these structures, it is necessary to know the conditions of strength and reliability and, in particular, to know the frequencies and the modes of natural vibrations in order to avoid the resonance frequencies, which may lead to the fracture of the entire structure. Hence, the reliable methods aimed at the numerical and experimental determination of the dynamic characteristics of these structures become especially important.

In the present work, we study the natural vibrations of a cantilevered fixed elliptic cylindrical shell of constant thickness. We note that number of works devoted to the problems of natural vibrations of noncircular cylindrical shells is quite small. In [2, 3, 9], the natural vibrations of elements of the shell structures and, in particular, of noncircular cylindrical shells, were studied by using numerical methods. The natural frequencies and the modes of vibration were also determined by the experimental methods. As one of these methods, we can mention the method of holographic interferometry allowing one to observe and measure the resonance frequencies and modes of vibration in the visible part of the analyzed surface with high accuracy and in real time [1, 7]. The analysis of the above-mentioned works enables us to conclude that there is no common viewpoint concerning the application of various approaches to the solution of the indicated class of problems.

The dynamics of shell structures is most often modeled within the framework of the classical Kirchhoff–Love theory with the use of numerical methods [4–6]. In the cases where the thickness varies or the shape of the middle surface is arbitrary, the displacements of points of these shells are described by a system of partial differential equations with variable coefficients. In this case, the variables cannot be separated with the help of Fourier series in one of the coordinate directions and the solution of the problems of natural vibrations of shells of variable thickness or with an arbitrary shape of the middle surface is accompanied by serious difficulties of the computational character.

At present, the so-called Computer-Aided Engineering (CAE) systems are extensively used for the solution of the problems of mechanics. These systems enable us to perform the numerical analyses of structures of any shape due to the application of the finite-element method, allow the user to estimate the behavior of the computerized model of a structure under the actual operating conditions, and to check the serviceability of the structure

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without significant losses of time and assets. As one of systems of this sort, we can mention a system based on an FEMAP pre- and post-processor equipped with an NX NASTRAN software module for the implementation of the engineering analyses by the finite-element method [8].

The aim of the present work is to study the natural vibrations of an elastic isotropic cylindrical shell with elliptic cross section by using the FEMAP program and to compare the accumulated numerical results with the experimentally obtained values.

Input Relations

We consider the problem of natural frequencies and modes of vibration. Under variable kinematic boundary conditions and in the absence of external actions, the dynamical equation takes the form

$$[M]\{\ddot{q}(t)\} + [C]\{\dot{q}(t)\} + [K]\{q(t)\} = 0, \tag{1}$$

where $[M]$ is the matrix of masses of the structure, $[C]$ is the matrix of coefficients of the forces of viscous damping, $[K]$ is the stiffness matrix, $\{q\}$ is the vector of nodal displacements and the overdot and two overdots in this equation denote, respectively, the first and second derivatives of q with respect to time.

In the absence of damping, Eq. (1) takes the form

$$[M]\{\ddot{q}(t)\} + [K]\{q(t)\} = 0. \tag{2}$$

The solution of the matrix equation (2) is

$$\{q(t)\} = \{A\}\cos(\omega t + \beta), \tag{3}$$

where $\{A\}$ is the vector of amplitudes of the nodal displacements, $\omega = 2\pi f$ is the circular frequency, and β is the phase of vibrations. As a result of the direct substitution of (3) in (2) and the reduction by $\cos(\omega t + \beta)$, we get the following system of algebraic equations:

$$\left(-\omega^2[M] + [K]\right)\{A\} = 0. \tag{4}$$

In this system, the nonzero values of the components of $\{A\}$ are possible only under the condition

$$\det[[K] - \omega^2[M]] = 0. \tag{5}$$

If the square matrices $[M]$ and $[K]$ are positive definite, which is, as a rule, true for the problem of linear elasticity, then Eq. (5) has N nonnegative solutions (natural frequencies) ω_k [N is the number of unknowns in system (4)]. Moreover, double solutions are possible.

If condition (5) is satisfied, then one of Eqs. (4) is a consequence of the remaining equations. Therefore, each frequency ω_k corresponds to a certain ratio of the amplitudes A_{ki}. In other words, all amplitudes of the vector can be expressed via one of these amplitudes. The ratios of the amplitudes A_{ki} specify the k th natural mode of vibrations.

Equations (4) imply that all degrees of freedom are characterized by a synchronous motion in the course of vibrations with the natural frequency ω_k. Thus, the configuration of the structure does not change its basic form and we observe solely the variations of the amplitudes.

If we have N values of natural frequencies ω_k, then the solution of system (2) can be sought in the form of a linear combination of N expressions (3):

$$\{q(t)\} = \sum_{k=1}^{N} \{A_k\} \cos(\omega_k t + \beta_k). \tag{6}$$

Thus, the time variations of the shape of a linear elastic structure are given as linear combinations of all its eigenshapes.

Note that the values of components of the eigenvectors $\{A_k\}$ can be found to within a constant factor. Hence, they are usually normalized according to the rule

$$\{A_k\}^{\top}[M]\{A_k\} = 1. \tag{7}$$

It is proved that the eigenvectors $\{A_k\}$ are orthogonal with respect to the matrices $[M]$ and $[K]$, i.e.,

$$\{A_k\}^{\top}[M]\{A_m\} = 0, \qquad \{A_k\}^{\top}[K]\{A_m\} = 0, \quad k \neq m. \tag{8}$$

We also note that, most often, it suffices to find only several smallest roots of Eq. (5) (denoted by N_α) and the other roots are not determined because the amplitudes of vibrations have relatively large values only for the lowest natural frequencies. In other words, several first natural frequencies and modes of vibration are sufficient to get a satisfactory approximation to solution (6). Several algorithms are developed for this purpose. The best of these algorithms are realized in the NX Nastran software module [8].

Procedure of Solution

With the help of the FEMAP program, we constructed the geometry of the shell in the form of a cylindrical surface with elliptic cross section of the following sizes: height $h = 120$ mm, the major semiaxis $a = 51.8$ mm, and the minor semiaxis $b = 37.295$ mm.

The parameters of modeling were chosen in correspondence with the shape and sizes of the shell for which the natural frequencies and the modes of vibration were studied by the method of holographic interferometry (Fig. 1). Shell 1 was manufactured by turning together with the massive body 2 in order to realize a rigid fastening along the contour in the course of experiments. Since the method of holographic interferometry reveals the absence of displacements of the baluster, it is reasonable to model the shell in calculations performed with the help of FEMAP as cantilever fastened along one of the contours.

The shell was made of 40Kh steel with the following parameters: Young's modulus $E = 214$ GPa, Poisson's ratio $\nu = 0.26$, and density $\rho = 7820$ kg/m^3. The partition was realized by linear quadrangular *plate*-elements 1mm × 1mm in size with constant thickness $d = 2$ mm and contained 33,396 nodes and 33,120 elements (Fig. 2). The fastening of cantilever was realized along the lower contour of the shell.

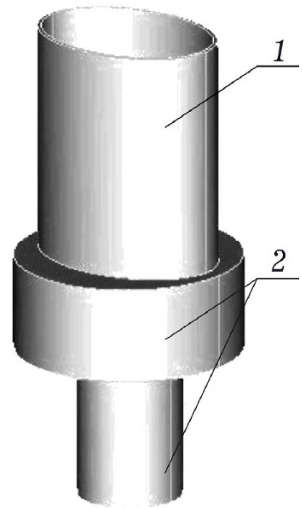


Fig. 1

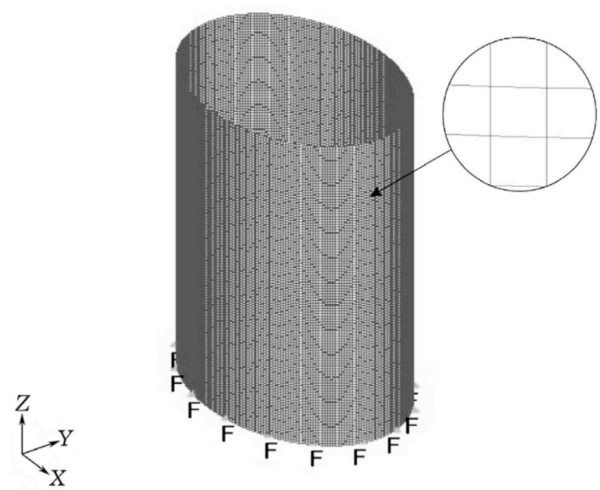


Fig. 2

Further, we performed the analysis of frequencies and modes of natural vibrations with the help of the NX NASTRAN software module.

For the sake of comparison, the shell was also specified as a bulky body with partition into *solid*-elements, which had absolutely no influence on the frequencies and modes of vibration.

We also performed the experimental investigations of the analyzed object on the base of the educational and scientific Laboratory of Holographic Methods at the Chair of Mathematics and Mechanics of the Sukhomlinskii Nikolaev National University. To perform the experiments and observe the interferograms of the excited shell in real time, we assembled a special optical scheme [1]. The shell made of high-alloyed steel with the same characteristics as the characteristics of the shell used for numerical modeling was manufactured by turning on a lathe equipped with the CNC.

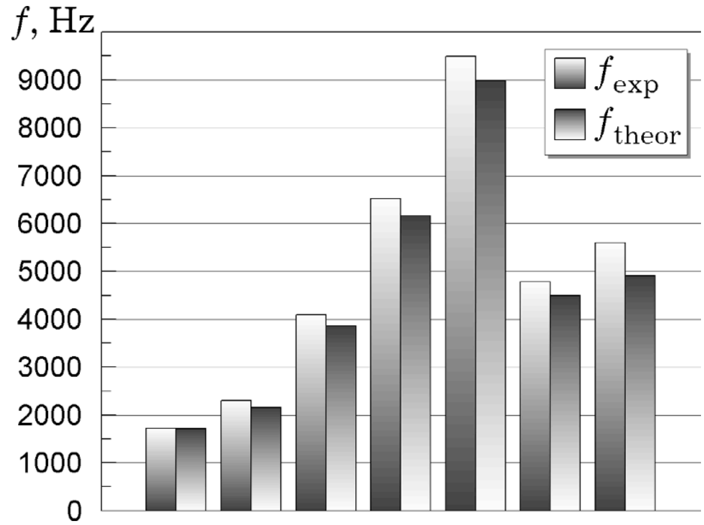


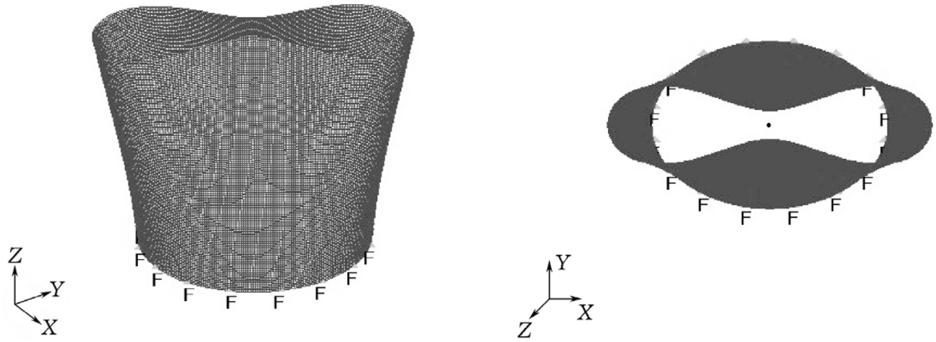
Fig. 3

Table 1

m	n	f_{exp}, Hz	f_{theor}, Hz	ε, %
1	4	1723	1712	0.64
1	6	2306	2161	6.29
1	8	4097	3860	5.78
1	10	6519	6154	5.60
1	12	9493	8978	5.43
2	6	4782	4498	5.94
2	8	5597	4911	12.26

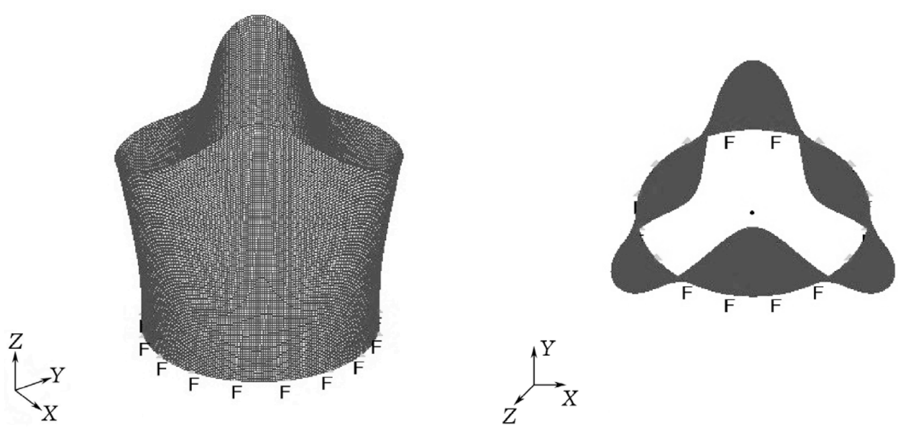
Analysis of the Results

The proposed method was used to study the spectrum of resonance frequencies and modes of vibration for a cantilever fastened isotropic shell of constant thickness. For comparison, the natural frequencies of vibrations were also found experimentally. The resonance frequencies are presented in the form of a histogram in Fig. 3 and in Table 1, where m is the number of nodes along the generatrix and n is the number of nodes along the circumferential coordinate. The modes of vibration at some frequencies are shown in Fig. 4 for two different directions. In this case, the displacements of points of the shell are displayed with tenfold magnification for the purposes of visualization.



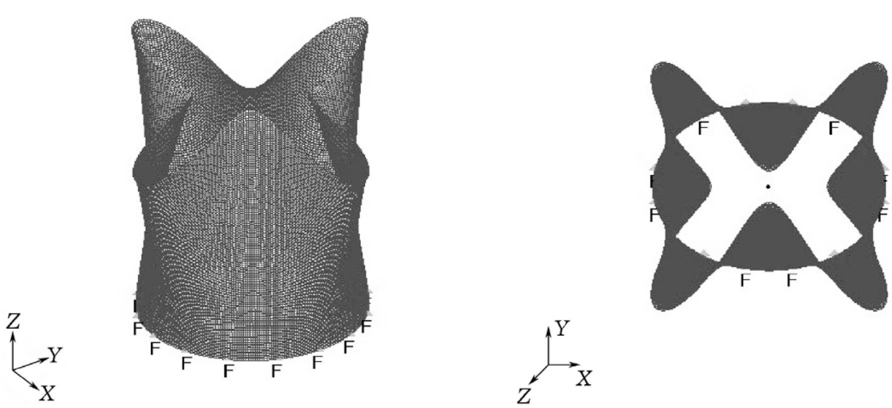
(a) (b)

Mode 1: $m = 1$, $n = 4$, $f = 1712$ Hz.



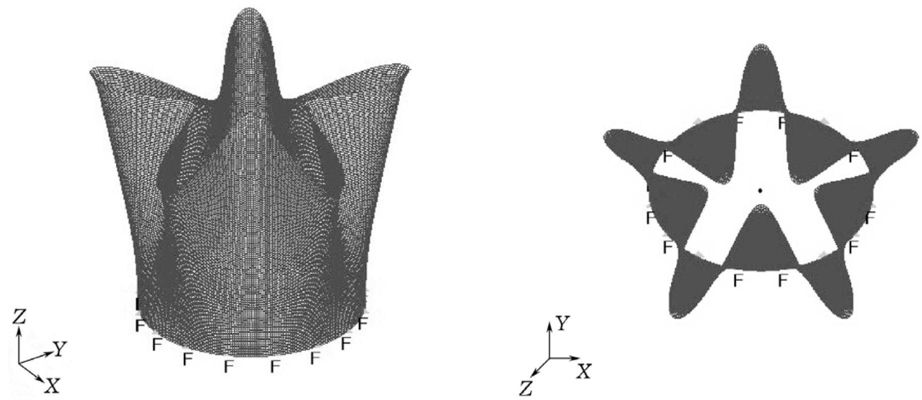
(a) (b)

Mode 3: $m = 1$, $n = 6$, $f = 2161$ Hz.



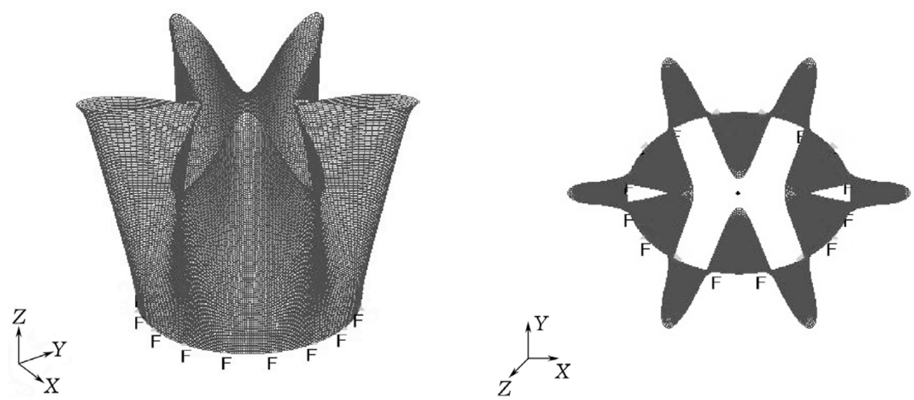
(a) (b)

Mode 6: $m = 1$, $n = 8$, $f = 3860$ Hz.



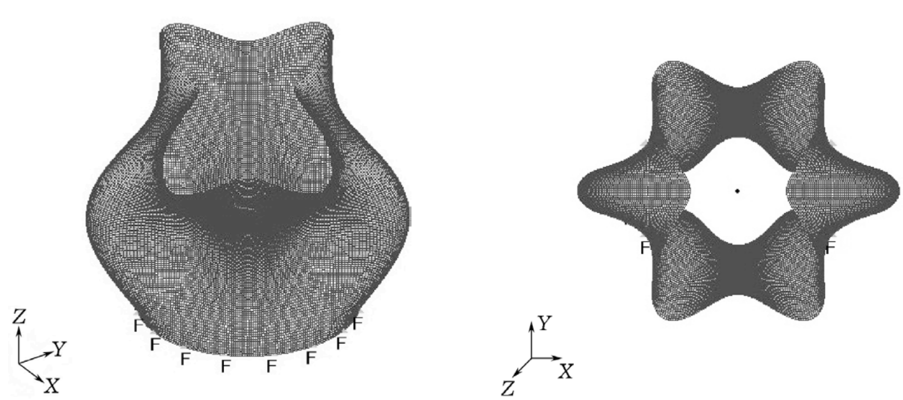
(a) **(b)**

Mode 13: $m = 1$, $n = 10$, $f = 6154$ Hz.



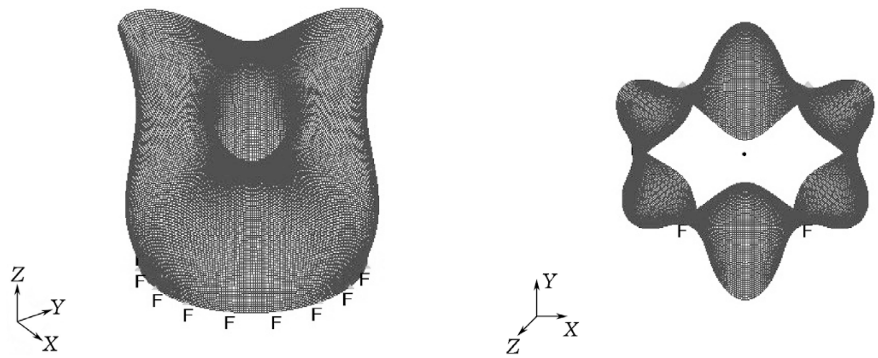
(a) **(b)**

Mode 25: $m = 1$, $n = 12$, $f = 8978$ Hz.



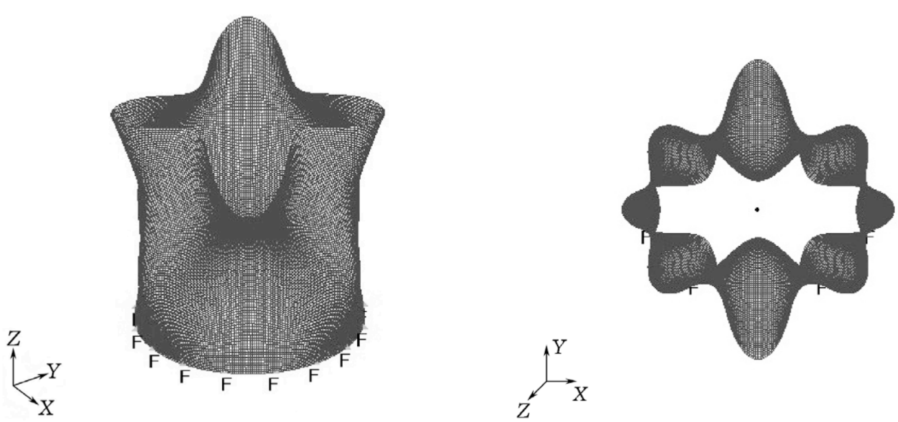
(a) **(b)**

Mode 15: $m = 2$, $n = 4$, $f = 6169$ Hz.



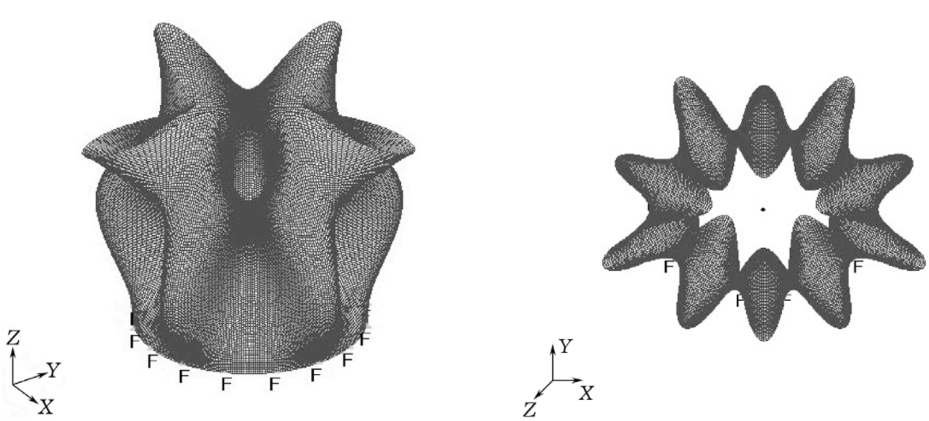
(a) (b)

Mode 9: $m = 2$, $n = 6$, $f = 4498$ Hz.



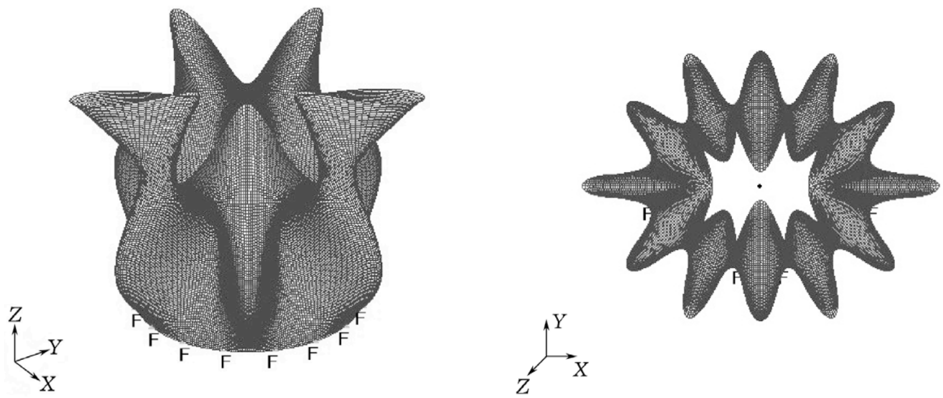
(a) (b)

Mode 11: $m = 2$, $n = 8$, $f = 4911$ Hz.



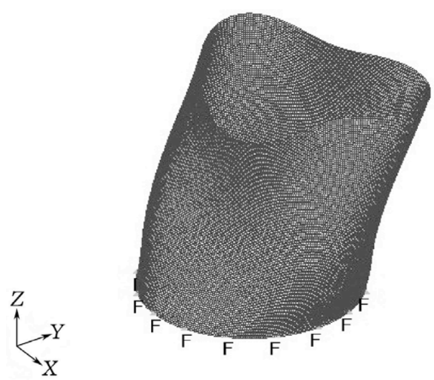
(a) (b)

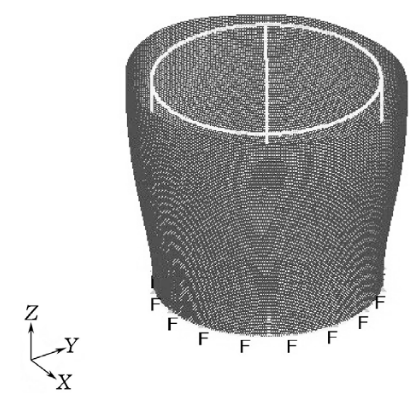
Mode 18: $m = 2$, $n = 10$, $f = 7099$ Hz.



(a) **(b)**

Mode 29: $m = 2$, $n = 12$, $f = 9772$ Hz.





(a) **(b)**

Mode 8: $m = 1$, $n = 6$ (bending), $f = 4104$ Hz. **Mode 17:** $m = 1$, $n = 0$ (breathing), $f = 6712$ Hz.

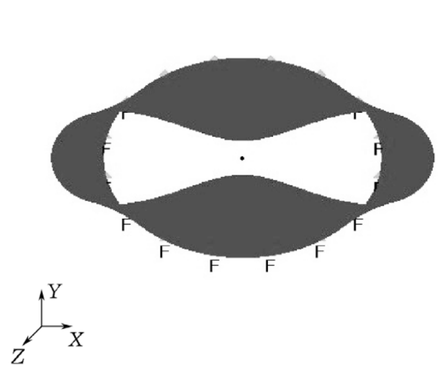
Fig. 4

The study performed with the help of the FEMAP program was realized for 60 modes within the range of up to 17 кHz. The analysis of the results reveal the repetition of the modes with symmetry breaking relative to the minor semiaxis for close frequencies. This was especially well visible for the modes with the number of nodes along the generatrix $m = 1$ (Fig. 5).

CONCLUSIONS

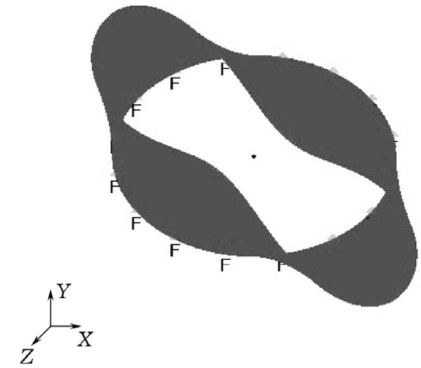
The procedure of modeling of the shell cantilever fastened along one of the contours of its cylindrical surface with elliptic cross section fairly reliably reflects the actual conditions of the experiment.

The experimental frequencies and the frequencies computed with the help of the NASTRAN module differ from by at most 13 %. For some frequencies, the difference is less than 1 %.



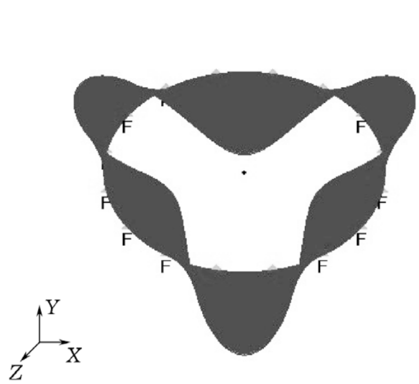
(a)

Mode 1: $f = 1712$ Hz.



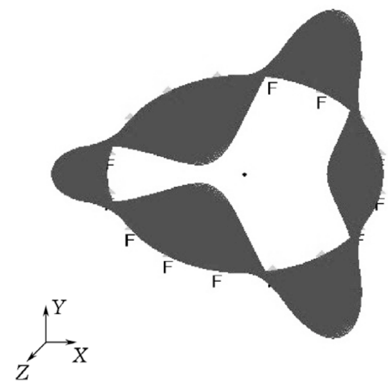
(b)

Mode 2: $f = 1761$ Hz.



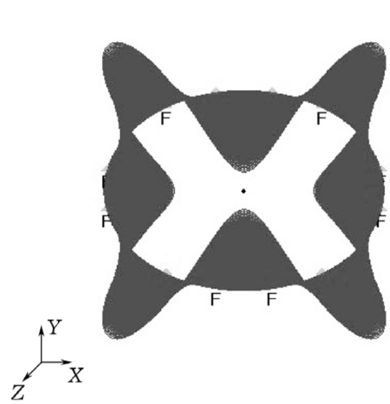
(a)

Mode 3: $f = 2161$ Hz.



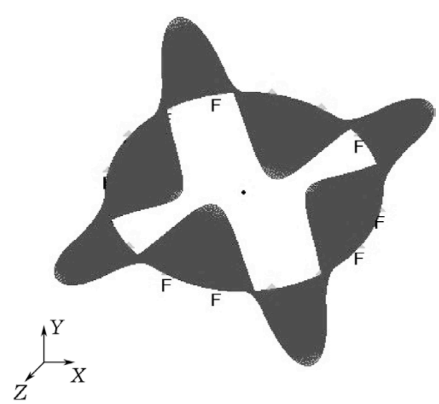
(b)

Mode 4: $f = 2167$ Hz.



(a)

Mode 6: $f = 3860$ Hz.



(b)

Mode 7: $f = 3863$ Hz.

Fig. 5

Unlike the experimental method, the numerical modeling allows us to get the complete spectrum of the modes of vibration. Indeed, in the experiments, the results depend on the conditions of excitation, accuracy of measuring devices, and the quality of the experiment.

The fact that the experimental and theoretical values of frequencies are in fairly good agreement confirms high reliability of both methods of investigations.

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