ALGORITHM TO DETERMINE THE OPTIMAL PARAMETERS OF A POLYNOMIAL WIENER FILTER–EXTRAPOLATOR FOR NONSTATIONARY STOCHASTIC PROCESSES OBSERVED WITH ERRORS

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Abstract. *The apparatus of canonical expansions of stochastic processes is used to obtain an algorithm to determine the optimal parameters of a discrete polynomial Wiener filter–extrapolator for nonstationary stochastic processes with errors.*

Keywords: *stochastic process, optimal extrapolation algorithm.*

Let a stochastic process $X(t)$ being analyzed in a discrete series of points t_i, $i=\overline{1,I}$, be measured with an error $Y(i)$, which results in a stochastic process of measurements $Z(t)$:

$$Z(t) = X(t) + Y(t). \tag{1}$$

Suppose that its first $k<1$ values are obtained: $Z(\mu) = z(\mu)$, $\mu = \overline{1,k}$. The extrapolation problem is to use this information and a priori information about the processes $X(t), Y(t)$ to obtain the optimal (in the mean-square sense) estimate $\hat{X}(i)$, $i=\overline{k+1,I}$, of the future values of the corresponding realization of the unobserved stochastic process $X(t)$.

One of the best known methods to solve this problem is the Wiener method [1, 2], according to which the estimate of the further values $\hat{X}(i)$, $i=\overline{k+1,I}$, of the realization of the stochastic process $X(t)$ can be found from

$$\hat{X}(i) = \sum_{\mu=1}^{k} h_\mu(i) z(\mu), \quad i=\overline{k+1,I}. \tag{2}$$

In (2) $h_\mu(i)$, $\mu=\overline{1,k}$, is a discretized Dirac response; its optimal values can be determined from the system of equations

$$\sum_{\mu=1}^{k} h_\mu(i) R_z(j,\mu) = R_{xz}(j,i), \quad j=\overline{1,k}, \quad i=\overline{k+1,I}. \tag{3}$$

The result of the forecast when the Wiener method is used is unbiased and minimizes the extrapolation mean-square error. However, the application domain of this algorithm is essentially bounded since obtaining it involves the assumptions that the stochastic process under study $X(t)$ and the error measurement process $Y(t)$ are stationary. In [3], this constraint is removed and the optimal (in the mean-square sense) solution of the problem of filtration–extrapolation of the nonstationary stochastic process is obtained:

$$m_{x/z}^{(\mu)}(i) = \begin{cases} 0, & \mu=0,\ i=\overline{1,I}, \\ m_{x/z}^{(\mu-1)}(i) + [z(\mu) - m_{x/z}^{(\mu-1)}(\mu)]\beta_\mu(i), & \mu=\overline{1,k},\ i=\overline{\mu+1,I}. \end{cases} \tag{4}$$

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Algorithm (4) has an equivalent explicit notation:

$$m_{x/z}^{(k)}(i) = \sum_{\mu=1}^{k} z(\mu) b_{\mu}^{(k)}(i), \quad i = \overline{k+1, I}, \tag{5}$$

where

$$b_{\mu}^{(k)}(i) = \begin{cases} b_{\mu}^{(k-1)}(i) - b_{\mu}^{(k-1)}(k) B_k(i), & \mu \le k-1; \\ \beta_k(i), & \mu = k. \end{cases} \tag{6}$$

The parameters of the algorithm (4), (5) are elements of canonical expansion [4, 5] of the mixed stochastic sequence $\{X'\} = \{Z(1), Z(2), \ldots, Z(k), X(k+1), \ldots, X(I)\}$, which combines both the measurement results up to $i = k$ and data on the process $X(t)$ for $i = \overline{k+1, I}$:

$$X'(i) = \sum_{\nu=1}^{i} U_{\nu} \beta_{\nu}(i), \quad i = \overline{1, I}; \tag{7}$$

$$U_i = Z(i) - \sum_{\nu=1}^{i-1} U_{\nu} \beta_{\nu}(i), \quad i = \overline{1, k}; \tag{8}$$

$$U_i = X(i) - \sum_{\nu=1}^{i-1} U_{\nu} \beta_{\nu}(i), \quad i = \overline{k+1, I}; \tag{9}$$

$$D_i = D_z(i) - \sum_{\nu=1}^{i-1} D_{\nu} \beta_{\nu}^2(i), \quad i = \overline{1, k}; \tag{10}$$

$$D_i = D_x(i) - \sum_{\nu=1}^{i-1} D_{\nu} \beta_{\nu}^2(i), \quad i = \overline{k+1, I}; \tag{11}$$

$$\beta_{\nu}(i) = \frac{1}{D_{\nu}} \{R_z(\nu, i) - \sum_{j=1}^{\nu-1} D_j \beta_j(\nu) \beta_j(i)\}, \quad \nu = \overline{1, k}, \quad i = \overline{\nu, k}; \tag{12}$$

$$\beta_{\nu}(i) = \frac{1}{D_{\nu}} \{R_{zx}(\nu, i) - \sum_{j=1}^{\nu-1} D_j \beta_j(\nu) \beta_j(i)\}, \quad \nu = \overline{1, k}, \quad i = \overline{k+1, I}; \tag{13}$$

$$\beta_{\nu}(i) = \frac{1}{D_{\nu}} \left\{ R_x(\nu, i) - \sum_{j=1}^{\nu-1} D_j \beta_j(\nu) \beta_j(i) \right\}, \quad \nu = \overline{k+1, I}, \quad i = \overline{k+1, I}. \tag{14}$$

A significant shortcoming of the algorithm (4), (5) is the assumption that there are only correlation relationships in the stochastic process $X(t)$.

The Wiener polynomial algorithm allows taking into account higher-order stochastic relationships in the solution of the forecast problem:

$$\hat{X}(i) = \sum_{j=1}^{k} \sum_{\nu=1}^{N} h_j^{(\nu)}(i) z^{\nu}(j), \quad \nu = \overline{1, N}, \quad i = \overline{k+1, I}. \tag{15}$$

The optimal values of the discretized Dirac response $h_j^{(\nu)}(i)$, $j = \overline{1, k}$, $\nu = \overline{1, N}$, $i = \overline{k+1, I}$, can be found from the system of equations

$$\sum_{j=1}^{k} \sum_{\nu=1}^{N} h_j^{(\nu)}(i) M[Z^{\nu}(j) Z^{l}(\mu)] = M[Z^{l}(\mu) X(i)],$$

$$j, \mu = \overline{1, k}, \quad \nu, l = \overline{1, N}, \quad i = \overline{k+1, I}. \tag{16}$$

However, method (15), as well as the filter–extrapolator (2), is applied only to stationary stochastic processes. This brings up a problem of removing this constraint for the nonlinear case.

PROBLEM STATEMENT

Let the stochastic process $X(t)$ under study be completely specified in a discrete series of points t_i, $i=\overline{1,I}$, by a descretized moment function $M[X^{\lambda}(\nu)X^{h}(i)]$, $\lambda,h=\overline{1,N}$, $\nu,i=\overline{1,I}$. The stochastic properties of $M[Y^{\lambda}(\nu)Y^{h}(i)]$, $\lambda,h=\overline{1,N}$, $\nu,i=\overline{1,I}$, of the stochastic process $Y(t)$ of measurement errors are also known. Without loss of generality, suppose $M[X(i)]=0$ and $M[Y(i)]=0$, $i=\overline{1,I}$. It is necessary to obtain the optimal (in the mean-square sense) algorithm of the extrapolation of the further values of the process under study $X(t)$ based on the results of successive measurements $z(\mu)$, $\mu=\overline{1,k}$.

SOLUTION METHOD

This problem can be solved based on the corresponding polynomial expansion [6, 7] of the sequence $\{X'\}$:

$$X'(i)=\sum_{\nu=1}^{i}\sum_{\lambda=1}^{N}W_{\nu}^{(\lambda)}\beta_{1\nu}^{(\lambda)}(i),\ \ i=\overline{1,I}. \tag{17}$$

The elements of expansion (17) are defined by the following recurrent relations:

$$W_{\nu}^{(\lambda)}=Z^{\lambda}(\nu)-\sum_{\mu=1}^{\nu-1}\sum_{j=1}^{N}W_{\mu}^{(j)}\beta_{\lambda\mu}^{(j)}(\nu)-\sum_{j=1}^{\lambda-1}W_{\nu}^{(j)}\beta_{\lambda\nu}^{(j)}(\nu),\ \ \nu=\overline{1,k}; \tag{18}$$

$$W_{\nu}^{(\lambda)}=X^{\lambda}(\nu)-\sum_{\mu=1}^{\nu-1}\sum_{j=1}^{N-1}W_{\mu}^{(j)}\beta_{\lambda\mu}^{(j)}(\nu)-\sum_{j=1}^{\lambda-1}W_{\nu}^{(j)}\beta_{\lambda\nu}^{(j)}(\nu),\ \ \nu=\overline{k+1,I}; \tag{19}$$

$$D_{\lambda}(\nu)=M[\{W_{\nu}^{(\lambda)}\}^2]=M[Z^{2\lambda}(\nu)]-\sum_{\mu=1}^{\nu-1}\sum_{j=1}^{N}D_j(\mu)\{\beta_{\lambda\mu}^{(j)}(\nu)\}^2$$

$$-\sum_{j=1}^{\lambda-1}D_j(\nu)\{\beta_{\lambda\nu}^{(j)}(\nu)\}^2,\ \ \nu=\overline{1,k}; \tag{20}$$

$$D_{\lambda}(\nu)=M[\{W_{\nu}^{(\lambda)}\}^2]$$

$$=M[X^{2\lambda}(\nu)]-\sum_{\mu=1}^{\nu-1}\sum_{j=1}^{N}D_j(\mu)\{\beta_{\lambda\mu}^{(j)}(\nu)\}^2-\sum_{j=1}^{\lambda-1}D_j(\nu)\{\beta_{\lambda\nu}^{(j)}(\nu)\}^2,\ \ \nu=\overline{k+1,I}; \tag{21}$$

$$\beta_{h\nu}^{(\lambda)}(i)=\frac{M[W_{\nu}^{(\lambda)}Z^{h}(i)]}{M[\{W_{\nu}^{(\lambda)}\}^2]}=\frac{1}{D_{\lambda}(\nu)}\left\{M[Z^{\lambda}(\nu)Z^{h}(i)]\right.$$

$$\left.-\sum_{\mu=1}^{\nu-1}\sum_{j=1}^{N}D_j(\mu)\beta_{\lambda\mu}^{(j)}(\nu)\beta_{h\mu}^{(j)}(i)-\sum_{j=1}^{\lambda-1}D_j(\nu)\beta_{\lambda\nu}^{(j)}(\nu)\beta_{h\nu}^{(j)}(i)\right\}, \tag{22}$$

$$\lambda=\overline{1,h},\ 1\le\nu\le i\le k,$$

$$\beta_{h\nu}^{(\lambda)}(i)=\frac{M[W_{\nu}^{(\lambda)}X^{h}(i)]}{M[\{W_{\nu}^{(\lambda)}\}^2]}=\frac{1}{D_{\lambda}(\nu)}\left\{M[Z^{\lambda}(\nu)X^{h}(i)]\ -\ \sum_{\mu=1}^{\nu-1}\sum_{j=1}^{N}D_j(\mu)\beta_{\lambda\mu}^{(j)}(\nu)\beta_{h\mu}^{(j)}(i)-\sum_{j=1}^{\lambda-1}D_j(\nu)\beta_{\lambda\nu}^{(j)}(\nu)\beta_{h\nu}^{(j)}(i)\right\},$$

$$\lambda=\overline{1,h},\nu=\overline{1,k},i=\overline{k+1}; \tag{23}$$

$$\beta_{h\nu}^{(\lambda)}(i)=\frac{M[W_{\nu}^{(\lambda)}X^{h}(i)]}{M[\{W_{\nu}^{(\lambda)}\}^{2}]}=\frac{1}{D_{\lambda}(\nu)}\Bigg\{M[X^{\lambda}(\nu)X^{h}(i)]$$

$$-\sum_{\mu=1}^{\nu-1}\sum_{j=1}^{N}D_{j}(\mu)\beta_{\lambda\mu}^{(j)}(\nu)\beta_{h\mu}^{(j)}(i)-\sum_{j=1}^{\lambda-1}D_{j}(\nu)\beta_{\lambda\nu}^{(j)}(\nu)\beta_{h\nu}^{(j)}(i)\Bigg\}, \tag{24}$$

$$\lambda=\overline{1,h},\ k\le\nu\le i\le I.$$

In the canonical expansion (17), the stochastic process $X(t)$ is represented in the series of points under study t_i, $i=\overline{1,I}$, by N arrays $\{W^{(\lambda)}\}$, $\lambda=\overline{1,N}$, of noncorrelated centered stochastic coefficients $W_i^{(\lambda)}$, $i=\overline{1,I}$. These coefficients contain the information on the values of $Z^{\lambda}(i)$, $\lambda=\overline{1,N}$, $i=\overline{1,k}$, and $X^{\lambda}(i)$, $\lambda=\overline{1,N}$, $i=\overline{k+1,I}$, and the coordinate functions $\beta_{h\nu}^{(\lambda)}(i)$, λ, $h=\overline{1,N}$, ν, $i=\overline{1,I}$, describe the probabilistic constraints of order $\lambda+h$ between sections t_ν and t_i, ν, $i=\overline{1,I}$.

Assume that the value $z(1)$ of the sequence $\{X'\}$ at the point t_1 becomes known as a result of measurement. Hence, the values of the coefficients $W_1^{(\lambda)}$, $\lambda=\overline{1,N}$, are known:

$$w_1^{(\lambda)}=z^{\lambda}(1)-M[Z^{\lambda}(1)]-\sum_{j=1}^{\lambda-1}w_1^{(j)}\beta_{1\nu}^{(j)}(1),\quad \nu=\overline{1,I}. \tag{25}$$

Substituting $w_1^{(1)}$ into (17) yields the polynomial canonical expansion of the a posteriori stochastic sequence $\{X'^{(1,1)}\}=X'(i/z_1(1))$:

$$X'^{(1,1)}(i)=X'(i/z(1))=z(1)\beta_{11}^{(1)}(i)+\sum_{\lambda=2}^{N}W_1^{(\lambda)}\beta_{11}^{(\lambda)}(i)+\sum_{\nu=2}^{i}\sum_{\lambda=1}^{N}W_\nu^{(\lambda)}\beta_{1\nu}^{(\lambda)}(i),\quad i=\overline{1,I}. \tag{26}$$

Applying the operation of expectation to (26) yields the optimal (in the criterion of the minimum mean-square extrapolation error) estimate of the future values of the sequence $\{X\}$ provided that one value $z(1)$ is used to determine this estimate:

$$m_{x/z}^{(1,1)}(i)=M[X'(i/z(1))]=z(1)\beta_{11}^{(1)}(i),\quad i=\overline{1,I}. \tag{27}$$

Since the coordinate functions $\beta_{h\nu}^{(\lambda)}(i)$, $\lambda,h=\overline{1,N}$, $\nu,i=\overline{1,I}$, are determined from the condition of the minimum mean-square approximation error within the intervals between arbitrary values of $Z^{\lambda}(\nu)$ and $X^{h}(i)$, expression (27) can be generalized to the case of forecasting $x^{h}(i)$, $h=1,\overline{N}$:

$$m_{x/z}^{(1,1)}(h,i)=M[X^{h}(i/z(1))]=z(1)\beta_{h1}^{(1)}(i),\quad i=\overline{k+1,I}. \tag{28}$$

Specifying the second value $w_1^{(2)}$ in (26) yields the canonical expansion of the a posteriori sequence $\{X'^{(1,2)}\}=X(i/z_1(1),z_1(1)^2)$:

$$X'^{(1,2)}(i)=X(i/z(1),z(1)^2)=z(1)\beta_{11}^{(1)}(i)+[z^2(1)-z(1)\beta_{21}^{(1)}(1)]\beta_{11}^{(2)}(1)$$

$$+\sum_{\lambda=3}^{N-1}W_1^{(\lambda)}\beta_{11}^{(\lambda)}(i)+\sum_{\nu=2}^{i}\sum_{\lambda=1}^{N-1}W_\nu^{(\lambda)}\beta_{1\nu}^{(\lambda)}(i),\quad i=\overline{1,I}. \tag{29}$$

Applying the operation of expectation to (29) and using expression (28) yield the algorithm of extrapolation based on two values, $z_1(1)$ and $z_1(1)^2$:

$$m_{x/z}^{(1,2)}(h,i)=M[X^{h}(i/z(1),z(1)^2)]=m_{x/z}^{(1,1)}(h,i)+[z^2(1)-m_{x/z}^{(1,1)}(2,i)]\beta_{11}^{(2)}(1),\quad i=\overline{k+1,I}. \tag{30}$$

Generalizing the obtained pattern allows writing the forecasting algorithm for an arbitrary number of known values:

$$m_x^{(\mu,l)}(h,i)=\begin{cases} m_x^{(\mu,l-1)}(h,i)+(z^{h}(\mu)-m_x^{(\mu,l-1)}(l,\mu))\beta_{h\mu}^{(l)}(i), & l\ne 1,\\ m_x^{(\mu,N-1)}(h,i)+(z^{h}(\mu)-m_x^{(\mu-1,N-1)}(l,\mu))\beta_{h\mu}^{(1)}(i), & l=1.\end{cases} \tag{31}$$

The expression $m_{x/z}^{(\mu,l)}(h,i)=M[X^h(i)/z^\nu(j),\ j=\overline{1,\mu-1},\ \nu=\overline{1,N};\ z^\nu(\mu),\ \nu=\overline{1,l}]$ for $h=1$, $l=N$, and $\mu=k$ is an unbiased optimal estimate $m_{x/z}^{(k,N-1)}(1,i)$ of the future value $x(i)$, $i=\overline{k+1,I}$, provided that the values of $z^\nu(j)$, $\nu=\overline{1,N}$, $j=\overline{1,k}$ are used to find this estimate, i.e., the results of the measurements of the sequence $\{X'\}$ at the points t_j, $j=\overline{1,k}$, are known.

The unknown estimate $m_{x/z}^{(k,N)}(1,i)$ can be written as

$$m_{x/z}^{(k,N)}(1,i)=\sum_{j=1}^{k}\sum_{\nu=1}^{N} z^\nu(j)S_{(j-1)N+\nu}^{(kN)}((i-1)N+1),\quad i=\overline{k+1,I}, \tag{32}$$

where

$$S_{(j-1)N+\nu}^{(\alpha)}(\xi)=\begin{cases} S_{(j-1)N+\nu}^{(\alpha-1)}(\xi)-S_{(j-1)N+\nu}^{(\alpha-1)}(\alpha)\beta_{\mathrm{mod}_N(\xi),j}^{(\nu)}(i), & \alpha-1\le(j-1)N+\nu; \\ \beta_{\mathrm{mod}_N(\xi),j}^{(\nu)}([\xi/N]+1), & \alpha=(j-1)N+\nu,\ \{\xi/N\}\ne 0; \\ \beta_{\mathrm{mod}_N(\xi),j}^{(\nu)}([\xi/N]), & \alpha=(j-1)N+\nu,\ \{\xi/N\}=0. \end{cases} \tag{33}$$

The mean square error of the extrapolation is defined by the expression

$$M[X(i/z^\nu(j),\ \nu=\overline{1,N},\ j=\overline{1,k})-m_x^{(k,N)}(1,i)]=M[X^2(i)]-\sum_{j=1}^{k}\sum_{\nu=1}^{N}M[(W_j^{(\nu)})^2](\beta_{1j}^{(\nu)}(i))^2,\ i=\overline{k+1,I}. \tag{34}$$

PROOF OF THE EQUIVALENCE

If the predicted process $X(t)$ and the process of measurement errors $Y(t)$ are stationary, the prediction algorithm (15) and (32) coincide.

To relate $h_j^{(\nu)}(i)$ with $S_{(j-1)N+\nu}^{(kN)}((i-1)N+1)$, let us consider in detail the mechanism of forming the optimal values of $h_j^{(\nu)}(i)$.

If one measurement $z(1)$ is used for the forecast, the optimal value of the coefficient $h_1^{(1)}(i)$ is determined according to (16) from the expression

$$h_1^{(1)}(i)=\frac{M[Z(1)X(i)]}{M[Z^2(1)]}.$$

In view of the properties of elements of the canonical expansion (17) and relation (33), to search for the values of the weight coefficients we have $h_1^{(1)}(i)=\beta_{11}^{(1)}(i)=S_1^{(1)}(i)$.

If two values are used for the forecast, $z(1)$ and $z^2(1)$, the system of equations (16) becomes

$$\begin{cases} h_1^{(1)}(i)M[Z(1)Z(1)]+h_1^{(2)}(i)M[Z^2(1)Z(1)]=M[Z(1)X(i)], \\ h_1^{(1)}(i)M[Z(1)Z^2(1)]+h_1^{(2)}(i)M[Z^2(1)Z^2(1)]=M[Z^2(1)X(i)]. \end{cases}$$

Using the Gauss method, we obtain

$$\begin{cases} h_1^{(1)}(i)+h_1^{(2)}(i)\dfrac{M[Z^3(1)]}{M[Z^2(1)]}=\dfrac{M[Z(1)X(i)]}{M[Z^2(1)]}, \\ h_1^{(2)}(i)(M[Z^4(1)-\dfrac{M^2[Z^3(1)]}{M[Z^2(1)]})=M[Z^2(1)X(i)]-\dfrac{M[Z(1)X(i)]M[Z^3(1)]}{M[Z^2(1)]}. \end{cases} \tag{35}$$

Considering the properties of elements of the canonical expansion (17), we represent the system of equations (35) as

$$\begin{cases} h_1^{(1)}(i)+h_1^{(2)}(i)\beta_{21}^{(1)}(1)=\beta_{11}^{(1)}(i), \\ h_1^{(2)}(i)=\beta_{11}^{(2)}(i). \end{cases}$$

The system yields the expressions for the optimal values of $h_1^{(1)}(i)$ and $h_1^{(2)}(i)$:

$$h_1^{(1)}(i)=\beta_{11}^{(1)}(i)-\beta_{11}^{(2)}(i)\beta_{21}^{(1)}(1),\ \ h_1^{(2)}(i)=\beta_{11}^{(2)}(i).$$

These expressions, as well as in the case of one value $z(1)$ used for the prediction, coincide with the expressions for the weight coefficients of algorithm (32).

Thus, for $N=2$ and $k=1$, $h_1^{(1)}(i)=S_1^{(2)}(2i-1)$, $h_1^{(2)}(i)=S_2^{(2)}(2i-1)$.

Continuing the reasoning for increasing *N* and *k*, it is easy to verify that algorithm (32) and the Wiener algorithm coincide for stationary stochastic processes.

CONCLUSIONS

We have obtained an algorithm to find the optimal values of a polynomial Wiener filter–extrapolator for nonstationary stochastic processes observed with errors. The solution is universal since the canonical expansion (17) exists and describes exactly, at discreteness points, any stochastic process with finite variance. Determining the optimal Dirac responses based on (33) is much easier as compared with the formation and solution of the systems of equations (16) for a large number of measurements and degree of nonlinearity.

REFERENCES

1. N. Wiener, Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications, The MIT Press, Cambridge (Mass.), Wiley and Sons, New York (1949).
2. Ya. P. Dragan, Models of Signals in Linear Systems [in Russian], Naukova Dumka, Kyiv (1972).
3. V. D. Kudritskii and I. P. Atamanyuk, “Algorithm to determine the optimal parameters of a Wiener filter–extrapolator for nonstationary stochastic processes observed with errors,” Cybern. Syst. Analysis, **32**, No. 3, 466–470 (1996).
4. V. S. Pugachev, Theory of Random Functions and its Application [in Russian], Fizmatgiz, Moscow (1962).
5. V. D. Kudritskii, Predicting the Reliability of Radioelectronic Devices [in Russian], Tekhnika, Kyiv (1982).
6. I. P. Atamanyuk, “Vector polynominal canonical expansion of a stochastic process,” Tekhn. Diagnostika i Nerazrush. Kontrol’, No. 1, 40–43 (2003).
7. I. P. Atamanyuk, “Algorithm of extrapolation of a nonlinear random process on the basis of its canonical decomposition,” Cybern. Syst. Analysis, **41**, No. 2, 267–273 (2005).