ТЕХНІЧНІ НАУКИ

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FINE ELASTIC CIRCULAR INCLUSION IN THE AREA OF HARMONIC VIBRATIONS OF AN UNLIMITED BODY UNDER SMOOTH CONTACT

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The problem of the interaction of harmonic waves with a thin elastic circular inclusion, which is located in an elastic isotropic body (matrix), is solved. On both sides of the inclusion between it and the body (matrix), the conditions of smooth contact are realized. The solution method is based on representing the displacements in the matrix through discontinuous solutions of the Lamé equations for harmonic vibrations. This made it possible to reduce the problem to Fredholm integral equations of the second kind with respect to functions associated with jumps in normal stress and radial displacement to included ones. After the realization of the boundary conditions on the sides of the inclusion, a system of singular integral equations is obtained to determine these jumps.

Keywords*: elastic inclusions, cylindrical waves, matrix, stress intensity factor.*

Formulation of the problem. Modern problems of dynamic fracture mechanics, improvement of means of non-destructive testing and flaw detection require further development and improvement of methods for solving problems of dynamic interaction of thin-walled inclusions with the environment. An important case of inclusions is a circular (disc-shaped) inclusion. This is primarily due to the fact that thin disc-shaped reinforcements are quite common in machine parts and building structures. Thin inclusions are not only stress concentrators, but are also used as fillers in composites. When creating composite materials, the matrix is often filled with coin-like reinforcing elements of high rigidity. Therefore, they are the inclusions of this shape that have always been given a lot of attention, which requires the solution of problems on the stress-strain state of bodies with inhomogeneities such as thin inclusions.

When solving problems on the vibrations of elastic bodies containing thin inclusions, it is often assumed that the inclusion is absolutely rigid. This assumption greatly facilitates the mathematical solution, but on the other hand does not allow taking into account the influence of the elastic properties of the inclusion on the stress concentration near it. The fact that this influence can be significant was shown in [1], where oscillations of an unbounded body with strip inclusions were considered. Oscillations of bodies with inclusions having low rigidity were considered in [2], [3]. In the present work, a method based on the use of discontinuous solutions is used to solve the problem of harmonic vibrations of a body with an inclusion in the form of a circular elastic plate.

Analysis of current research. An unlimited elastic body (matrix) in which there is an inclusion in the form of an elastic disk is considered, thickness h i radius $a\left(h << a\right)$. If you enter a cylindrical coordinate system, than in the area $z=0$ it occupies a circle $r \le a, \quad 0 \le \theta < 2\pi$. The inclusions are under the action of waves propagating in the matrix. Several cases of wave action are considered. In the first case, a flat longitudinal wave propagates in the medium, the front of which is parallel to the plane of inclusion. This wave is given by the potential and causes displacement in the matrix

$$\varphi_0 = \frac{A_0 e^{i\kappa_1 z}}{\kappa_1}, \quad u_z^0 = iA_0 e^{i\kappa_1 z}, \qquad u_r^0 \equiv 0. \qquad (1)$$

In the second case, the matrix propagates cylindrical waves of expansion-compression, the potential and displacement caused by these waves are determined by the formulas [4]:

$$\varphi_0(r,z) = \frac{A_0}{\beta_1} J_0(\beta_1 r) e^{i\gamma z};$$

$$u_z^0 = \frac{i\gamma A_0}{\beta_1} J_0(\beta_1 r) e^{i\gamma z}; \qquad u_r^0 = -A_0 J_1(\beta_1 r) e^{i\gamma z}. \quad (2)$$

The third case is the interaction with the inclusion of a cylindrical shear wave with potential [4] and causes in the moving medium:

$$\psi_0(r,z) = \frac{B_0}{\beta_2^2} J_0(\beta_2 r) e^{i\gamma z},$$

$$u_z^0 = B_0 J_0(\beta_2 r) e^{i\gamma z}, \quad u_r^0 = -\frac{i\gamma B_0}{\beta_2} J_1(\beta_2 r) e^{i\gamma z}. \qquad (3)$$

In formulas (1) - (3) the notation is accepted:

$$\kappa_k = \frac{\omega}{c_k}, \quad \beta_k = \sqrt{\kappa_k^2 - \gamma^2}, \quad k = 1,2;$$

$$c_1^2 = \frac{\lambda_1 + 2\mu_1}{\rho_1}, \quad c_2^2 = \frac{\mu_1}{\rho_1}. \qquad (4)$$

where λ_1, μ_1 – constant Lame matrix, ρ_1 – matrix density. Multiplier $e^{-i\omega t}$, which determines the dependence on time here and further discarded.

The conditions of interaction of the inclusion with the matrix on the basis of the small thickness of the inclusion are formulated relative to its median area. At smooth contact on inclusion normal pressure and radial movement for which jumps designations are entered will be discontinuous

$$<\sigma_z> = \sigma_z(r,+0) - \sigma_z(r,+0) = \chi_1(r);$$

$$<u_z> = u_r(r,+0) - u_r(r,+0) = \chi_4(r). \qquad (5)$$

Equality must be observed on both sides of the inclusion

$$\tau_{zr}(r,\pm 0) = 0, \quad u_z(r,\pm 0) = w_0(r),$$

$$0 \le r < a. \qquad (6)$$

Here $w_0(r)$ – bend the displacement of the median plane of inclusion, which is determined from the equation of bending oscillations of round plates [5] under conditions of axial symmetry

$$D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 w_0 - m\omega^2 w_0 = \chi_1(r),$$

$$0 \le r < a, \qquad (7)$$

where $D = \frac{E_0 h^3}{12(1-\nu_0^2)}$ – cylindrical stiffness, $m = h\rho_0$ - mass per unit area of inclusion. Equations (10) are considered with free-edge conditions

$M(a) = 0$; $Q(a) = 0$,

where $M(r)$ – bending moment and $Q(r)$ – transverse force included. From the last equations it follows that

$$\left.\left(\frac{\partial^2 w_0}{\partial r^2} + \frac{\nu_0}{r}\frac{\partial w_0}{\partial r}\right)\right|_{r=a} = 0;$$

$$\left.\frac{\partial}{\partial r}\left(\frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\frac{\partial w_0}{\partial r}\right)\right|_{r=a} = 0, \qquad 0 \le r < a \quad . \quad (8)$$

In addition, the function $w_0(r)$ should be limited when $r \to 0$.

The aim of the article is to solve the problem of harmonic oscillations of an unbounded body with a disc-shaped inclusion in the case of smooth contact conditions**.** Previously, this problem was solved for a completely rigid inclusion. [6].

Presenting main material. To solve the boundary value problem (7), (8), the Green's function is first constructed, which is determined by the formulas:

$$G_1(\eta, r) = g_1(\eta, r) -$$

$$-\frac{1}{q_{01}}\left(J_0(q_{01}\eta)G_1(r) + I_0(q_{01}\eta)G_2(r)\right). \qquad (9)$$

In the formula (9) $g_1(\eta, r)$ – the fundamental function of equation (10) which is equal to:

$$g_1(\eta, r) = \frac{1}{2q_1}\left(g_1^-(\eta, r) + g_1^+(\eta, r)\right);$$

$$g_1^{\pm}(\eta, r) = \int_0^{\infty} \frac{\lambda J_0(\lambda r) J_0(\lambda \eta)}{\lambda^2 \pm q_1^2} d\eta.$$

For other functions included in (9) there are equalities:

$$G_1(r) = \frac{1}{2\Delta(q_{01})}\left(A_1(q_{01}) J_0(q_1 r) + \frac{I_0(q_1 r)}{q_{01}}\right);$$

$$G_2(r) = \frac{1}{2\Delta(q_{01})}\left(A_2(q_{01}) J_0(q_1 r) + \frac{I_0(q_1 r)}{q_{01}}\right);$$

$$A_1(q_{01}) = I_1(q_{01}) a_2(q_{01}) + \frac{\pi i}{2} h_1(q_{01}) H_1^{(1)}(q_{01});$$

$$A_2(q_{01}) = a_1(q_{01}) K_1(q_{01}) - J_1(q_{01}) h_2(q_{01});$$

$$\Delta(q_{01}) = a_1(q_{01}) I_1(q_{01}) + J_1(q_{01}) h_1(q_{01});$$

$$a_1(q_{01}) = J_0(q_{01}) - (1 - \nu_0)\frac{J_1(q_{01})}{q_{01}};$$

$$a_2(q_{01})=\frac{\pi i}{2}\left(H_0^{(1)}(q_{01})-(1-\nu_0)\frac{H_1^{(1)}(q_{01})}{q_{01}}\right);$$

$$h_1(q_{01})=I_0(q_{01})-(1-\nu_0)\frac{I_1(q_{01})}{q_{01}};$$

$$h_2(q_{01})=K_0(q_{01})+(1-\nu_0)\frac{K_1(q_{01})}{q_{01}};$$

$$q_1^4=\frac{m\omega^2}{D};\qquad q_{01}=qa.$$

Using the Green's function, the solution of the boundary value problem (7), (8) is given in the form

$$w_0(r)=\int_0^a\frac{\chi_1(\eta)}{D}G_1(\eta,r)d\eta,\quad 0\le r<a. \quad (10)$$

To determine the displacements and stresses in the matrix, which are included in condition (5), they are presented as

$$u_z=u_z^0+u_z^1,\quad \tau_{rz}=\tau_{rz}^0+\tau_{rz}^1. \quad (11)$$

In these formulas u_z^0, τ_{rz}^0 – displacements and stresses are caused in the medium by a propagating wave. Additions u_z^1, τ_{rz}^1 this displacement and stress are caused by waves reflected from the inclusion. They are represented by jumps (5) by means of a discontinuous solution of the Lamé equations, which for the case of oscillation of an elastic medium under conditions of axial symmetry [7] has the form:

$$u_z^1=\int_0^a\eta\frac{\chi_1(\eta)}{\mu_1}g_{31}(\eta,r,z)d\eta+\int_0^a\eta\chi_4(\eta)g_{34}(\eta,r,z)d\eta;$$

$$\tau_{rz}^1=\int_0^a\eta\chi_1(\eta)g_{21}(\eta,r,z)d\eta+$$

$$+\int_0^a\eta\chi_4(\eta)g_{24}(\eta,r,z)d\eta. \quad (12)$$

Then, after substituting (10), (11), (12) into boundary conditions (6), we obtain a system of integral equations with respect to unknown jumps.

$$\int_0^a\eta\chi_1(\eta)F_{21}(\eta,r)d\eta+\int_0^a\eta\chi_4^*(\eta)F_{24}(\eta,r)d\eta=-\frac{\tau_{rz}^0(r,0)}{\mu_1};$$

$$\int_0^a\eta\frac{\chi_1(\eta)}{\mu_1}F_{31}(\eta,r)d\eta+\int_0^a\eta\chi_4^*(\eta)F_{34}(\eta,r)d\eta=$$
$$=\int_0^a\frac{\chi_1(\eta)}{D}G_1(\eta,r)d\eta-u_z^0(r,0),\quad 0\le r<a. \quad (13)$$

When deriving the system (13) in the integrals that contain, was the integration of parts and introduced the notation:

$$\chi_4^*(\eta)=\frac{1}{\eta}\frac{d}{d\eta}(\eta\chi_4(\eta))$$

$$F_{21}(\eta,r)=\frac{1}{2\kappa_2^2}\int_0^\infty\frac{B(\lambda)}{q_2(\lambda)}\lambda^2J_0(\lambda\eta)J_1(\lambda r)d\lambda,$$

$$F_{24}(\eta,r)=\frac{\mu_1}{2\kappa_2^2}\int_0^\infty\frac{R(\lambda)}{q_2(\lambda)}\lambda J_1(\lambda\eta)J_1(\lambda r)d\lambda,$$

$$F_{31}(\eta,r)=-\frac{1}{2\mu_1\kappa_2^2}\int_0^\infty\frac{S(\lambda)}{q_2(\lambda)}\lambda J_0(\lambda\eta)J_0(\lambda r)d\lambda,$$

$$F_{34}(\eta,r)=-\frac{1}{2\kappa_2^2}\int_0^\infty\frac{B(\lambda)}{q_2(\lambda)}\lambda^2J_1(\lambda\eta)J_0(\lambda r)d\lambda,$$

where $q_1(\lambda)=\sqrt{\lambda^2-\kappa_1^2}$, $q_2(\lambda)=\sqrt{\lambda^2-\kappa_2^2}$.

In order to bring the system (13) to a form convenient for numerical solution and selection from the kernels of integral operators of the singular component over it, it is necessary to make transformations similar to those described in [8]. To do this, you must enter into consideration of the function

$$\begin{bmatrix}T_1(\lambda)\\T_4(\lambda)\end{bmatrix}=\int_0^a\eta\begin{bmatrix}\chi_1(\eta)\\\chi_4^*(\eta)\end{bmatrix}J_0(\lambda\eta)d\eta.$$

Then the first and second equations of the resulting system must be acted upon by operators

$$D_1[f]=\frac{d}{dx}\int_0^x\frac{ydy}{\sqrt{x^2-y^2}}\int_0^y f(r)dr,$$

$$D_2[f]=\frac{d}{dx}\int_0^x\frac{rf(r)}{\sqrt{x^2-y^2}}dr.$$

Substitute the cosine representation by Fourier integrals into the equations obtained after this equation

$$\begin{bmatrix}T_1(\lambda)\\T_4(\lambda)\end{bmatrix}=\frac{2}{\pi}\int_0^a\begin{bmatrix}\varphi_1(\tau)\\\varphi_4(\tau)\end{bmatrix}\cos\lambda\tau d\tau.$$

Functions $\varphi_k(\tau)$ associated with jumping $\chi_k(\eta)$, equations

$$\chi_1(\eta)=\frac{2}{\pi}\int_\eta^a\varphi_1(\tau)\frac{d}{d\tau}\left(\frac{1}{\sqrt{\tau^2-\eta^2}}\right)d\eta;$$

$$\chi_4^*(\eta)=-\frac{2}{\pi}\int_\eta^a\varphi_2(\tau)\frac{d}{d\eta}\left(\frac{1}{\sqrt{\tau^2-\eta^2}}\right)d\tau;$$

$$\varphi_1(\tau)=\int_\tau^a \frac{\eta\chi_1(\eta)}{\sqrt{\eta^2-\tau^2}}d\eta;\quad \varphi_2(\tau)=\int_\tau^a \frac{\tau\chi_4^*(\eta)}{\sqrt{\eta^2-\tau^2}}d\eta,\quad 0\le\tau<a,$$

$$\varphi_1(\tau)\equiv 0;\quad \varphi_2(\tau)\equiv 0,\quad \tau>a.$$

As a result of these actions we find that functions $\varphi_k(\tau)$, $(k=1{,}4)$ are solutions of the following system of integral equations

$$\frac{1}{2\pi}\int_{-1}^{1} g_1(y)(R_1(y-\varsigma)-R_1(y))dy+$$

$$+\frac{1}{2\pi}\int_{-1}^{1} g_2(y)(R_2(y-\varsigma)-R_2(y))dy=f_1(\varsigma)$$

$$\frac{1}{2\pi}\int_{-1}^{1} g_1(y)R_3(y-\varsigma)dy+\frac{1}{2\pi}\int_{-1}^{1} g_2(y)R_1(y-\varsigma)dy= \quad (14)$$

$$=-\frac{1}{2\pi}\int_{-1}^{1} g_1(y)F(\zeta,y)dy+f_2(\varsigma).$$

When deriving equations (14), notations were also introduced

$$\tau=ay,\quad x=a\zeta,\quad \varphi_k(ay)=\mu_1 a g_k(y),\quad (k=1,2)$$

$$\kappa_0=a\kappa_2,\quad \frac{\gamma}{\kappa_2}=d_k,\ \xi=\frac{\kappa_1}{\kappa_2},\quad (k=1,2).\quad (15)$$

The right parts of the system (14) depending on the type of wave incident on the inclusion are determined by the following formulas. When interacting with a flat longitudinal wave

$$f_1(\xi)=0;\quad f_2(\zeta)=-i\alpha.$$

If cylindrical waves of expansion-compression propagate in the medium, then

$$f_1(\xi)=\frac{2i\alpha_0}{d_1}(1-\cos\xi\kappa_0 b_1 y);$$

$$f_2(\zeta)=\frac{d_1\alpha_0\cos(\xi\kappa_0 b_1 y)}{b_1}.$$

In the case of action on the inclusion of a cylindrical wave of transverse shear

$$f_1(\xi)=\frac{\beta_0(2b_2^2-1)(1-\cos\kappa_0 b_2 y)}{b_2^2};$$

$$f_2(\zeta)=\beta_0\cos(\kappa_0\beta_2 y).$$

In the resulting system functions $R_k(x)$, $(k=1,2,3)$ are determined by integrals

$$R_k(x)=\kappa_0\int_0^\infty B_k(u)\cos u\kappa_0 x du,\quad (k=1,2,3),\quad (16)$$

where $B_1(u)=\frac{ub(u)}{\sqrt{u^2-1}};\quad B_2(u)=\frac{R_0(u)}{u\sqrt{u^2-1}};$

$$B_3(u)=\frac{u\sigma(u)}{\sqrt{u^2-1}};$$

$$b(u)=2u^2-1-2\sqrt{u^2-1}\sqrt{u^2-\xi^2};$$

$$R_0(u)=(2u^2-1)^2-4u^2\sqrt{u^2-1}\sqrt{u^2-\xi^2}.$$

$$\sigma(u)=u^2-\sqrt{u^2-1}\sqrt{u^2-\xi^2},\quad u=\frac{\lambda}{\kappa_2}.$$

You can see that function $B_k(u)$, which are included in the integrals (16), are bounded by $u\to\infty$ and therefore these integrals must be understood in a generalized sense. To establish this value, we should use formulas (3.753) with [9] and formulas for differentiation of generalized functions [8]. As a result, we find:

$$R_k=R_{k1}+iR_{k2},\quad k=1,2,3.$$

$$R_{11}(p)=\frac{\pi}{2}[2C_{11}(p)\delta''(p)+2C_{12}(p)\delta'(p)+$$

$$+C_{13}(p)\delta(p)+C_{14}(p)sign(p)];$$

$$R_{21}(p)=\frac{\pi}{2}[4C_{21}(p)\delta''(p)+4C_{22}(p)\delta'(p)+$$

$$+C_{23}(p)\delta(p)+C_{24}(p)sign(p)];$$

$$R_{31}(p)=\frac{\pi}{2}[C_{31}(p)\delta''(p)+C_{32}(p)\delta'(p)+$$

$$+C_{33}(p)\delta(p)+C_{34}(p)sign(p)];$$

$$R_{21}(p)=\kappa_0(-2S_1(p)-S_2(p)+2\xi^2S_1(\xi p)+2\xi^3S_2(\xi p));$$

$$R_{22}(p)=\kappa_0(-4S_1(p)-4S_2(p)+$$

$$+4\xi^3S_1(p)+4\xi^3S_2(\xi p)-C_2(p));$$

$$R_{32}(p)=\kappa_0(2S_1(\kappa_0 p)-2\xi^3S_1(\xi\kappa_0 p)+\xi^3S_2(\xi\kappa_0 p));$$

where

$$\frac{1}{2}C_{11}(p)=\frac{1}{4}C_{21}(p)=-\frac{2}{\kappa_0^2}(J_0(\kappa_0 p)-J_0(\xi\kappa_0 p));$$

$$\frac{1}{2}C_{12}(p)=\frac{1}{4}C_{22}(p)=\frac{6}{\kappa_0}(J_1(\kappa_0 p)-\xi J_1(\xi\kappa_0 p));$$

$$C_{13}(p)=2(6A_1(p)-6\xi^2A_1(\xi p)+$$

$$+J_0(\kappa_0 p)-2\xi^3J_0(\xi\kappa_0 p));$$

$$C_{23}(p)=8(3A_1(p)-$$

$$-3\xi^2A_1(\xi p)-J_0(\kappa_0 p)+\xi^2J_0(\xi\kappa_0 p));$$

$$C_{14}(p)=\kappa_0(-2A_2(p)+2\xi^3A_2(\xi p)+$$

$$+J_1(\kappa_0 p)-2\xi^3J_1(\xi\kappa_0 p));$$

$$C_{24}(p) = -4\kappa_0\big(A_2(p) - \xi^3 A_2(\xi p) - J_1(\kappa_0 p) +$$

$$+\xi^3 J_1(\xi\kappa_0 p) - \kappa_0 C_1(p)\big);$$

$$C_{31}(p) = -\frac{2}{\kappa_0^2}\left(J_0(\kappa_0 p) - J_0(\xi\kappa_0 p)\right);$$

$$C_{32}(p) = \frac{6}{\kappa_0^2}\left(\kappa_0 J_1(\kappa_0 p) - \xi\kappa_0 J_1(\xi\kappa_0 p)\right);$$

$$C_{33}(p) = 6A_1(\kappa_0 p) - 6A_1\xi^2(\xi\kappa_0 p) + 2\xi^2 J_0(\xi\kappa_0 p);$$

$$C_{34}(p) = -\kappa_0 A_2(\kappa_0 p) + \kappa_0\xi^3 A_2(\xi\kappa_0 p) + \kappa_0\xi^3 J_1(\kappa_0 p);$$

$$A_1(x) = J_0(x) - \frac{J_1(x)}{x}; \quad A_2(x) = J_1(x) - \frac{2J_1(x)}{x^2} + \frac{J_0(x)}{x};$$

$$S_1(x) = \sum_{m=0}^{\infty} a_m x^{2m}; \quad S_2(x) = \sum_{m=0}^{\infty} b_m x^{2m};$$

$$a_0 = \frac{1}{3}; \quad b_0 = 1; \quad a_m = \frac{(-1)^m (m+1)}{(2m-1)!!(2m+1)!!};$$

$$b_m = \frac{(-1)^m}{(2m-1)!!(2m+1)!!}; \quad m = 1, 2, \ldots$$

$$C_1(p) = \sum_{k=0}^{\infty} d_k |\kappa_0 z|^{2k+1}; \qquad C_2(p) = \sum_{k=0}^{\infty} c_k (\kappa_0 z)^{2k+2};$$

$$d_k = \frac{(-1)^k}{(k!)^2 2^{2k}(2k+1)};$$

$$c_k = \frac{(-1)^k}{(2k+2)\big((2k+1)!!\big)^2}, \qquad (17)$$

$\delta(p)$ – delta Dirac function.

$$F(\zeta, y) = \frac{\xi_0}{\varepsilon^3 q_0^2}\Big[D^-\big(q_0(\zeta - y)\big) - D^+\big(q_0(\zeta - y)\big) -$$

$$-2\big(B_1(y)\cos q_0\zeta + B_2(y) ch q_0\zeta\big)\Big],$$

where

$$D^-(q_0|z|) = \big(\ln|z| - ci(q_0|z|)\big) + \big(1 - \cos q_0|z|\big) ci(q_0|z|) -$$

$$-\sin(q_0|z|)\left(si(q_0|z|) + \frac{\pi}{2}\right);$$

$$D^+(q_0|z|) = \big(1 - ch q_0|z|\big)\ln|z| - ch(q_0|z|)\big(C + \ln q_0|z|\big) -$$

$$-\frac{1}{2}\left(e^{-q_0|z|} S_3(q_0|z|) + e^{q_0|z|} S_3(-q_0|z|)\right),$$

$$S_3(x) = \sum_{k=1}^{\infty} \frac{x^k}{k\,k!};$$

Functions $B_k(y), k = 1,2$ are due to the action on the Green's function (10) and their derivatives by the differential operator and the calculation of the corresponding integrals, the introduction of notation (15) and the subtraction of the discontinuous component.

After substituting (17) into the system (14) and calculating the integrals with - function and its derivatives, we obtain a system of integral equations, the matrix record of which has the form

$$AG(y) + \frac{1}{2\pi}\int_{-1}^{1} Q(\zeta, y) G(\zeta) d\zeta +$$

$$+A_0 G(0) + \frac{1}{2\pi}\int_{-1}^{1} Q_0(\zeta) G(\zeta) d\zeta = F(y). \qquad (18)$$

Vectors $G(y)$, $F(y)$ and matrices A, A_0, Q_0 are by formulas

$$G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}; \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}; \quad A = \begin{pmatrix} -\frac{\xi^2}{2} & 1 - \xi^2 \\ -\frac{1+\xi^2}{4} & -\frac{\xi^2}{2} \end{pmatrix};$$

$$A_0 = \begin{pmatrix} \xi^2 & -(1 - \xi^2) \\ 0 & 0 \end{pmatrix}; \quad Q = \begin{pmatrix} -Q_1 & -Q_2 \\ -Q_3 & -Q_1 \end{pmatrix};$$

$$Q_0 = \begin{pmatrix} Q_1 & Q_2 \\ 0 & 0 \end{pmatrix};$$

$$Q_l(\zeta) = Q_{l1}(\zeta) + iQ_{l2}(\zeta), \quad l = 1,2$$

$$Q_{11}(p) = \frac{\pi\kappa_0}{2} sign(p) \times$$

$$\times\big(-2A_2(p) + 2\xi^3 A_2(\xi p) + J_1(\kappa_0 p) - 2\xi^3 J_1(\xi\kappa_0 p)\big);$$

$$Q_{21}(p) = \frac{\pi\kappa_0}{2} sign(p) \times$$

$$\times\Big[\big(A_2(p) - \xi^3 A_2(\xi p) - J_1(\kappa_0 p) + \xi^3 J_1(\xi\kappa_0 p) + C_1(p)\big)\Big]$$

$$Q_{31}(p) = \frac{\pi\kappa_0}{2} sign(p)\big(-A_2(p) + \xi^3 A_2(\xi p) + \xi^3 J_1(\xi\kappa_0 p)\big);$$

$$Q_{lj}(p) = R_{lj}(p), \quad l, j = 1,2.,$$

$$Q = \begin{pmatrix} -Q_1 & -Q_2 \\ -P_1 & -Q_3 \end{pmatrix},$$

$$P_1(\zeta, y) = Q_3(\zeta - y) - F(\zeta, y).$$

Because, $det(A) = \frac{1}{4} \neq 0$, then you can always find the inverse matrix for matrix A

$$A^{-1} = \begin{pmatrix} -2\xi^2 & -4(1 - \xi^2) \\ 1 + \xi^2 & -2\xi^2 \end{pmatrix}.$$

Multiply both parts of the system (18) by the matrix A^{-1} and enter the notation:

$$R = A^{-1}Q; \quad R^0 = A^{-1}Q_0; \quad D^0 = A^{-1}A_0;$$

As a result, the system (18) is transformed into a form

$$G(y)+\frac{1}{2\pi}\int_{-1}^{1}R(\zeta-y)G(\zeta)d\zeta+$$

$$+D^0G(0)+\frac{1}{2\pi}\int_{-1}^{1}R^0(\zeta)G(\zeta)d\zeta=$$

$$=H(y),-1\le y\le 1 \qquad (19)$$

Thus, system (19) is a system of Fredholm integral equations of the second kind, which allows effective approximate solutions.

The approximate solution of system (19) will be found in the form of an interpolation polynomial

$$G(y)=\sum_{m=1}^{n}G_m\frac{P_n(y)}{(y-y_m)P_n'(y_m)};$$

$$\mathrm{G_m}=\begin{pmatrix}g_1(y_m)\\ g_2(y_m)\end{pmatrix}.$$

$P_n(y)$ – polynomial Legendre, y_m, $(m=1,2...n)$ – his radical.

If we now use the Darboux-Christophel identity for Legendre polynomials [10], [11],

$$\frac{P_n(y)}{y-y_m}=-\frac{1}{n+1}\sum_{j=0}^{n-1}(2j+1)\frac{P_j(y)P_j(y_m)}{P_{n+1}(y_m)}$$

then we find that

$$G(0)=\sum_{m=1}^{n}A_m b_m^0 G_m;$$

$$b_m^0=\frac{1}{2}\sum_{j=0}^{n=1}(2j+1)P_j(0)P_j(y_m);$$

$$A_m=\frac{2}{\left(1-y_m^2\right)\left[P_m'(y_m)\right]^2}\,. \qquad (20)$$

We approximate the integrals in system (19) by the Gaussian quadrature formula [12] and use (20). As a result, we obtain the following system of linear algebraic equations for the approximate determination of the values of unknown functions in interpolation nodes:

$$G_j+\frac{1}{2\pi}\sum_{m=1}^{n}A_m\left[R(y_m-y_j)+R^0(y_m)+D^0b_m^0\right]G_m=F_j;$$

$$F_j=F(y_j), \quad j=1,2...m \qquad (21)$$

To estimate the stress concentration in the matrix near the inclusion is used, as in [8], [13] stress intensity coefficients (CIN), equal to:

$$K_1=\lim_{r\to a-0}\sqrt{a\text{-}r}\,\chi_1(r); \qquad K_3=\lim_{r\to a+0}\sqrt{a\text{-}r}\,\tau_{rz}^1(r,0).$$

After performing the boundary transition, we find that

$$K_j=\mu_1\sqrt{2a}N_j, \quad j=1,3;$$

$$N_1=\frac{g_1(1)}{\pi},$$

$$N_3=\frac{\xi^2}{2}g_1(1)-\left(1-\xi^2\right)g_2(1) \qquad (22)$$

Using (22), the dimensionless values of CIN are expressed through the solution of system (21) by formulas

$$N_1=\frac{\sigma_1}{2\pi}; \quad N_3=\frac{1}{2\pi}\left(\xi^2\sigma_1-2\left(1-\xi^2\right)\sigma_2\right);$$

$$\sigma_1=\sum_{m=1}^{n}C_m g_j(y_m);$$

$$C_m=\left((1-y_m)P_n'(y_m)\right)^{-1}. \qquad (23)$$

Consider the results of a numerical study on the frequency dependence of CIN. First, it was found that the rigidity of the inclusion on the stress concentration around it. For this purpose, it was assumed that the inclusion and the matrix have the same density and Poisson's ratios $(\overline{\rho}=1, \nu_0=\nu_1=0{,}25)$, the results of these studies can be seen in fig.1 a, b. The curves in these figures correspond to the specified value of the ratio of the modulus of elasticity of the matrix and the inclusion $e_0=\frac{E_1}{E_0}$ and that a flat longitudinal wave (1) interacts with the inclusion. Corresponding curves $e_0=10^{-5}$ on the fig. 1 i $e_0=10^{-4}$ on the fig. 1b completely coincide with similar constructions for absolutely rigid inclusion [6]. The behavior of both coefficients is complicated by a decrease in the stiffness of the inclusion and becomes complex with a large number of highs and lows.

Value $|N_1|$ for elastic inclusions, as a rule, exceed the corresponding values for absolutely rigid inclusions and this excess can reach several times. Value $|N_3|$ for elastic inclusion, as a rule, are smaller than those calculated for absolutely rigid inclusion. Calculations of CINs for inclusions and matrices from real materials were also performed. The results of these calculations illustrate the graphs in the fig. 1c-h. Solid curves are constructed taking into account the elasticity of the inclusion, and dotted according to the assumption that the inclusion is absolutely rigid. The inclusion was considered steel, and three types of materials were assumed for the matrix. Curves 1 are constructed for the concrete matrix, curves 2 are constructed under the assumption that the matrix material is lead, and curve 3 in fig. 1 c corresponds to a matrix of copper. The graphs in fig. 1c and fig. 1d show the change in CIN depending on the wave number when interacting with the inclusion of a flat longitudinal wave. The results of the calculations are shown in fig. 1e and fig.1f correspond to the case of the action on the inclusion of a longitudinal cylindrical wave, and those graphs that in fig. 1g and fig. 1h are constructed when a transverse cylindrical wave acts on the inclusion.

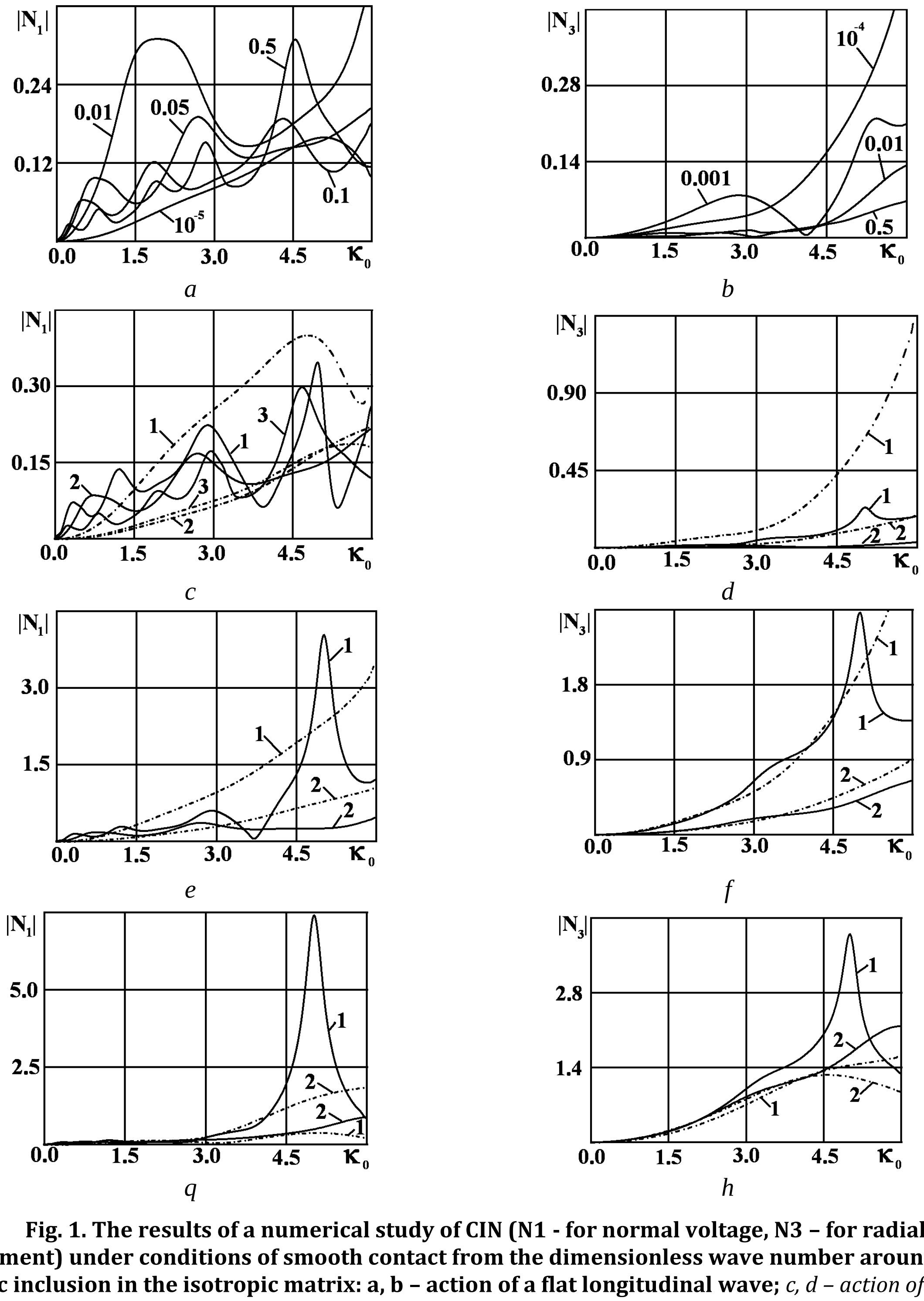

Fig. 1. The results of a numerical study of CIN (N1 - for normal voltage, N3 – for radial movement) under conditions of smooth contact from the dimensionless wave number around the elastic inclusion in the isotropic matrix: a, b – action of a flat longitudinal wave; *c, d – action of a flat longitudinal wave for real materials; e, f – action of a longitudinal cylindrical wave; g, h – the action of a transverse cylindrical wave*

Conclusions and prospects for further research

The analysis of research results indicates: 1. In the case of real materials, taking into account the elasticity of inclusions significantly affects the value of CIN. The values of CIN obtained taking into account the elasticity for some materials may exceed, and for some be much lower than those corresponding to the absolutely rigid inclusion. 2. Taking into account the stiffness of the inclusion also significantly changes the dependence of CIN on the wave number. It becomes more complex with many highs and lows. Moreover, the maximum values of CIN several times may exceed the corresponding values for absolutely rigid inclusions.

When calculating the strength of machine parts and structures, it is necessary to take into account the elasticity of inclusions.

Список використаних джерел:

1. Литвин О.В. Попов В.Г. Изгибные колебания тонкого упругого включения в неограниченной среде при взаимодействии с упругими волнами. *Теоретическая и прикладная механика.* 2002. Вип.38. С. 131-140.
2. Kit H.S., Kunets Ya. I., Yemets V.F. Elastodynamics scattering from a thinwallad inclusions of low rigidity. *International Journal of Engineering Science. 1999. v. 37. P. 331-343.*
3. Кит Г.С., Кунец В.В., Михаськив В.В. Взаимодействие стационарной волны с тонким дискообразным включением малой жесткости. *Известия РАН. Механика твердого тела. 2004. №5. С. 83-89.*
4. Михаськів В.В., Калиняк О.І. Нестаціонарні збурення тривимірної пружної матриці з жорстким дисковим включенням. *Фізико–хімічна механіка матеріалів*. 2005. Т. 41, № 2. С. 7-15.
5. Вахонина Л.В., Попов В.Г. Осесимметричные колебания пространства с тонким жестким круговым включением // Теория и практика процессов измельчения, разделения, смешения и уплотнения. Одесса: ОГМА, 2002. Вып. 9. С. 28-34.
6. Вахонина Л.В., Попов В.Г. Взаимодействие упругих волн с тонким жестким круговым включением в случае гладкого контакта. *Теоретическая и прикладная механика*. 2003. Вып. 38. С. 158-166.
7. Попов Г.Я. Построение разрывных решений дифференциальных уравнений теории упругости для слоистой среды с межфазными дефектами. *Доклады РАН*. 1999. Т.364, № 6. С.769-773.
8. Кит Г.С., Михаськив *В.В.,* Хай О.М. Анализ установившихся колебаний плоского абсолютно жесткого включения в трехмерном упругом теле методом граничных элементов. *Прикладная математика и механика*. 2002. Т. 66, Вып. 5. С. 855-863.
9. Градштейн И.С., Рыжик И.М. Таблицы интегралов, сумм рядов и произведений. М. , 2011. 1108с.
10. Rakhmanov Evguenii A. Orthogonal Polynomials Walter De Gruyter Incorporated. 2016. 510 c.
11. Суетин П.К. Класические ортогональние многочлены - 3-е изд., перераб. и доп. М.: Физмат, 2005. 480с.
12. Zemanian A.H. Distribution Theory and Transform Analysis: An Introduction to Generalized Functions, with Applications (Dover Books on Mathematics). 2010. 400 c.
13. Михаськів В.В., Кунець Я.І., Міщенко В.О. Напруження у тривимірному тілі з тонким податливим включенням за фронтом імпульсних хвиль. *Фізико–хімічна механіка матеріалів*. 2003. Т. 39, № 3. С. 63–68.

Л. В. Вахоніна, Н. В. Потриваєва, О. С. Садовий. **Тонке пружне кругове включення у зоні дії гармонічних коливань необмеженого тіла за умов гладкого контакту**

Розв'язана вісесиметрична задача про взаємодію гармонічних хвиль з тонким пружним круговим включенням, яке розташоване в пружному ізотропному тілі (матриці). На обох сторонах включення між ним та тілом (матрицею) реалізовані умови гладкого контакту. Метод розв'язання базується на поданні переміщень в матриці через розривні розв'язки рівнянь Ламе для гармонічних коливань. Це дозволило звести задачу до інтегральних рівнянь Фредгольма другого роду відносно функцій, зв'язаних зі стрибками нормального напруження і радіального переміщення на включенні. Після реалізації граничних умов на сторонах включення для визначення цих стрибків отримано систему сингулярних інтегральних рівнянь.

***Ключові слова**: пружне включення, циліндричні хвилі, матриця, коефіцієнт інтенсивності напружень.*

Л. В. Вахонина, Н. В. Потриваева, А. С. Садовой. **Тонкое упругое круговое включение в зоне действия гармонических колебаний неограниченного тела при гладком контакте**

Решена задача о взаимодействии гармонических волн с тонким упругим круговым включением, которое расположено в упругом изотропном теле (матрице). На обеих сторонах включения между ним и телом (матрицей) реализованы условия гладкого контакта. Метод решения базируется на представлении перемещений в матрице через разрывные решения уравнений Ламе для гармонических колебаний. Это *позволило свести задачу к интегральным уравнениям Фредгольма второго рода относительно функций, связанных со скачками нормального напряжения и радиального перемещения на включении. После реализации граничных условий на сторонах включения для определения этих скачков получена система сингулярных интегральных уравнений.*

***Ключевые слова**: упругое включение, цилиндрические волны, матрица, коэффициент интенсивности напряжений.*