Journal of Theoretical and Applied Mechanics, Sofia, Vol.52 (2022) pp. 164-178
FLUID MECHANICS

# STABILITY OF THE COUPLED LIQUID-ELASTIC BOTTOM OSCILLATIONS IN A RECTANGULAR TANK

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[Received: 09 April 2021. Accepted: 05 April 2022]

**doi: https://doi.org/10.55787/jtams.22.52.2.164**

ABSTRACT: Eigenoscillations of the elastic bottom of a rigid (two-dimensional) rectangular tank with an ideal incompressible liquid with irrotational flows, which completely fills it, are were investigated. The elastic bottom is a clamped thin rectangular plate subject to tensile or compressive forces in its middle surface. It is shown that the frequency equation is divided into two equations describing symmetric (even) and antisymmetric (odd) frequencies, and can be written in a single form for these frequencies. For even and odd frequencies, an approximate formula is obtained, from which approximate conditions follow for stability of coupled vibrations of an elastic basis and a liquid. Exact stability conditions are obtained. The stability conditions of the static approach coincide with the exact stability conditions of the dynamic approach. It is shown that the approximate value of the critical bending stiffness for asymmetric frequencies is 0.952 times lower, and for symmetric frequencies – 0.930 times.

KEY WORDS: hydroelasticity, elastic rectangular plate, ideal incompressible liquid, rectangular tank, flat oscillations, stability.

## 1 INTRODUCTION

This work is a continuation of article [1], in which the stability of normal vibrations of the elastic bottom of a rigid rectangular tank with an ideal incompressible liquid completely filling it was investigated. The elastic bottom was presented in the form of a membrane. In this article, the elastic bottom is a clamped thin rectangular plate subject to tensile or compressive forces in its middle surface. It is shown that when the plate degenerates into a membrane, we obtain the well-known results of the article [1]. The article [1] provides a fairly complete overview of publications on the problem under consideration at the end of the 20th and beginning of the 21st centuries. Below is an overview of only the main publications of recent years.

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In article [2] we propose the construction of approximate solutions to two-dimensional hydroelastic problems that describe free oscillations of an ideal liquid in a horizontally placed long cylindrical container with an arbitrary symmetric cross-section. The free surface of the liquid is covered with a plane membrane or an elastic plate. Using concrete examples, we analyze the obtained solutions and the results of the calculations of frequencies, and the form of oscillations of the mechanical system under consideration.

Plane vibrations of a rectangular plate horizontally separating ideal incompressible liquids of different densities in a rigid rectangular channel with elastic bases were considered in [3]. The elastic top and bottom base are provided in the form of rectangular plates. A similar problem is discussed in the article [4] for a rigid rectangular channel with rigid bases. It is shown that for the case of clamped, supported and free contours, the frequency equation breaks down into two equations describing even and odd frequencies, and can be written in a single form for these frequencies. If the contours of the plate have different fixation, then the frequency equation is not divided into even and odd frequencies. The greatest simplification of the frequency equation was achieved in the case of clamped contours. In this case, the previously obtained approximate conditions for the stability of oscillations of the plate and liquid are refined.

The approach developed in paper [5] is applied to vibration analysis of rectangular plates coupled with liquid. The plates can be totally submerged in liquid or floating on its free surface. The mathematical model for the structure is developed using a combination of the finite element method and Sanders' shell theory. The article [6] deals with the study of the behavior of an idealized two-dimensional hydroelastic system involving two inviscid liquids with an elastic rectangular container. The main objective is to study the influence of the physical parameters on the system eigenfrequencies and eigenmodes. The governing equations describing the behavior of the system are analyzed using the concept of normal modes and their solutions presented in the form of infinite series. The expansion coefficients for the velocity potentials are calculated using a new inner product that allows the orthogonalization of the normal modes. The paper [7] deals with a theoretical dynamic model of the fuel assembly submerged in the coolant, and presents a free vibration analysis of a bundle of identical rectangular plates fully in contact with an ideal liquid. The orthogonal polynomial functions, as admissible functions, were generated using the Gram–Schmidt process to approximate the wet dynamic displacements of the plates with a clamped-clamped-free-free boundary condition. The natural frequencies under the wet condition were calculated using the Rayleigh–Ritz method based on minimizing the Rayleigh quotient of the ratio between the maximum potential energy and total kinetic energy. The comparison showed the excellent agreement between the results from the proposed theoretical method with the finite element analysis results.

In [8] the hydrostatic vibration analysis of a laminated composite rectangular plate partially contacting with a bounded liquid is studied. Wet dynamic transverse displacements of the plate are approximated by a set of admissible trial functions which are required to satisfy the clamped and simply supported geometric boundary conditions. Natural frequencies of the plate coupled with sloshing liquid modes are calculated using Rayleigh–Ritz method based on minimizing the Rayleigh quotient. The proposed analytical method is validated with available data in the literature.

The article [9] presents an analytical solution for free vibration analysis of thick rectangular isotropic plates coupled with a bounded liquid for various boundary conditions. In order to consider displacement theories of an arbitrary order, the Carrera Unified Formulation (CUF) is used. The eigenvalue problem is obtained by using the energy functional, considering plate and liquid kinetic energies as well as the potential energy of the plate. The Ritz method is used to evaluate the displacement variables, and the functions used in the Ritz series can be adjusted to consider arbitrary vibration with the classical boundary conditions. The convergence of the solution is analyzed, and the validation of results considering open literature and the 3D finite element software is performed.

Many works deal with hydroelastic oscillations of an ideal liquid in circular and coaxial cylinders with rigid and elastic bases, i.e., [10–16] and many others. In [10], the problem of free vibrations of an ideal liquid in a container in the form of a right circular cylinder with an arbitrary axisymmetric bottom is considered in the case when the unperturbed free surface of the liquid is covered with an elastic membrane or a plate. Using the expansion in terms of eigenfunctions of an auxiliary spectral problem with a parameter in the boundary conditions and the method of decomposition of the region of the meridional section of the container, an analytical solution to the problem is obtained. These solutions are analyzed and frequencies and modes of vibration are calculated. In article [11] derives the virtual masses and frequencies for asymmetric free vibration of the coupled system including a clamped circular plate in contact with incompressible bounded liquid. Considering small oscillations induced by the plate vibration in an incompressible and inviscid liquid, velocity potential function is used to describe the liquid motion. Derivation uses the Kirchhoff's thin plate theory. Two approaches are used to derive the free vibrations frequencies of the system. The solutions include an analytical solution employing the Fourier-Bessel series and a variational formulation applied simultaneously to the plate and liquid. The strong correlation between free vibrations frequencies of two solutions is found. Finally, the effect of liq uid depth on the virtual masses and free vibration frequencies of the coupled system is studied.

The paper [12] deals with the hydroelastic vibration of a circular elastic diaphragm interacting with the incompressible and inviscid liquid inside the cylindrical chamber with a central discharge opening. Taking into account axisymmetric vibration of the diaphragm, the liquid pressure exerted upon the plate is formulated using the linear Bernoulli's equation. Numerical results are presented for different materials for diaphragm (silicon and glass) and pumped liquid (water and methanol). Normal frequencies of the coupled system, wet mode shapes of diaphragm, and liquid oscillation modes are presented using numerical simulations. It is seen that the hydroelastic interaction lowers considerably natural frequencies, however the wet mode shapes for diaphragm vibration are very similar to the dry mode shapes.

In the article [13], the frequency equation of axisymmetric oscillations of a heavy two-layer ideal liquid in a rigid annular cylindrical tank with an elastic top and bottom in the form of clamped annular plates is derived and researched. The work [14] deals with the study of frequency equations of asymmetric and symmetric natural oscillations of an ideal bilayer liquid in a rigid circular cylindrical tank with an elastic top and bottom in the form of clamped circular plates. Using the example of a homogeneous liquid with a free surface and an elastic bottom in the form of a membrane, the frequency spectrum was analyzed analytically and numerically.

Numerical study of the effects of a thin plate covering a cylindrical rigid fuel tank filled with an inviscid, irrotational, and incompressible fluid is given in [15]. Governing equations of liquid motion coupled by plate vibration are solved analytically. We study the effect of a parameter on the natural frequency of coupled liquid-structure interaction.

The semianalytical scheme is proposed to explore the effect of the single flexible baffle on the coupled responses in the rigid cylindrical tank partially filled with an ideal liquid undergoing the pitching excitation [16]. The function series for the velocity potential, the dynamic deflection of the flexible baffle, and the surface wave height are given by introducing the time-dependent generalized coordinates. The Stokes–Joukowski potentials which are contained in the liquid velocity potential can be solved analytically. According to the dynamic and kinematic equations for the free surface and the coupled vibration equation for the flexible baffle, the coupled dynamic response equations are obtained. The additional damping terms are introduced to account the sloshing damping. The semianalytical method is validated by the comparison with the numerical results.

In this paper, we investigate the normal vibrations of the elastic bottom of a rigid rectangular tank with an ideal incompressible liquid that completely fills it. The elastic bottom is a clamped thin rectangular plate subject to tensile or compressive forces in its middle surface. It is shown that the frequency equation is separated into two equations describing symmetric (even) and antisymmetric (odd) frequencies, and can be written for these frequencies in a single form.

This article researches the normal vibrations of the elastic bottom of a rigid rectangular tank with an ideal incompressible liquid that completely fills it. It is shown that the frequency equation is divided into two equations describing symmetric and antisymmetric frequencies. For these frequencies, an approximate formula is obtained, from which the approximate conditions for the stability of coupled vibrations of an elastic foundation and a liquid follows. Exact stability conditions are obtained. The stability conditions of the static approach coincide with the exact stability conditions of the dynamic approach.

## 2 Formulation of the Problem

Let us consider small coupled plane vibrations of the elastic bottom of a rigid rectangular tank with an ideal and incompressible liquid. The elastic bottom is presented in the form of a thin rectangular plate with flexural rigidity $D$, subjected to tensile ($T > 0$) or compressive ($T < 0$) forces in the median surface. The rectangular tank width $b$ ($b = 2a$) and height $h$ is filled with a liquid with density $\rho$. The rectangular tank of width $b$ ($b = 2a$) and height $h$ is filled with a liquid of density $\rho$. The coordinate system $Oxyz$ is such that the plane $Oxy$ lies on the undisturbed surface of the plate, the axis $Oy$ is directed along its middle of the tank, and the axis $Oz$ is opposite to the acceleration vector of gravity $\bar{g}$ (Fig. 1).

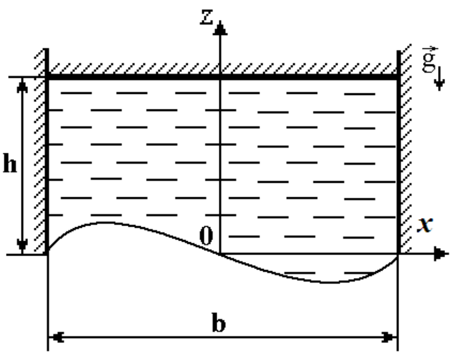


Fig. 1: A rigid rectangular tank with a liquid and an elastic bottom in the form of a plate.

Oscillations of the plate and liquid are considered in a linear formulation, assuming that their vibrations are inseparable, and the motion of the liquid is potential. Equations of plane vibrations of an elastic plate and a liquid have the form [1,4]

$$k_0 \frac{\partial^2 W}{\partial t^2} + D\frac{\partial^4 W}{\partial x^4} - T\frac{\partial^2 W}{\partial x^2} - g\rho W = -\sum_{n=1}^{\infty} \frac{a_n \ddot{W}_n}{k_n}\psi_n + Q + P_0, \quad (1)$$

$$\int_{-a}^{a} W dx = 0, \qquad W_n = \frac{1}{N_n^2} \int_{-a}^{a} W \psi_n dx$$

with boundary conditions

$$W\Big|_{x=\pm a} = 0, \qquad \frac{dW}{dx}\Big|_{x=\pm a} = 0. \tag{2}$$

Here $k_0 = \rho_0 h_0$; $W(x,t)$, $\rho_0$, $h_0$ are, respectively, normal flexure, density and thickness of the plate; $a_n = \rho \coth \kappa_n$, $\kappa_n = hk_n$, $k_n = \pi n/b$, $\psi_n(x) = \cos k_n(x+a)$, $N_n^2 = \int_{-a}^{a} \psi_n^2 dx = a$; $Q$ is an arbitrary time function, $P_0$ is a preset external pressure on the elastic bottom.

## 3 Normal Coupled Oscillations of the Elastic Plate and a Liquid

To find the eigenfrequencies of the coupled oscillations of the elastic plate and the liquid, we set

$$W(x,t) = w(x)\, e^{i\omega t}, \qquad Q = \tilde{Q} e^{i\omega t}, \qquad P_0 = \tilde{P}_0 e^{i\omega t}.$$

Considering (1) and (2), we obtain the boundary value problem

$$\frac{d^4 w}{dx^4} - p\frac{d^2 w}{dx^2} - qw = \frac{\omega^2}{D} \sum_{n=1}^{\infty} \frac{a_n w_n}{k_n} \psi_n + C, \tag{3}$$

$$w_n = \frac{1}{a} \int_{-a}^{a} w \psi_n dx, \qquad \int_{-a}^{a} w dx = 0, \tag{4}$$

$$w\Big|_{x=\pm a} = 0, \qquad \frac{dw}{dx}\Big|_{x=\pm a} = 0, \tag{5}$$

where $p = T/D$, $q = (k_0\omega^2 + g\rho)/D > 0$, $C = (\tilde{Q} + \tilde{P}_0)/D$.

We look for the solution of the Eq. (3) in the form of the general solution of the homogeneous equation and a particular solution of the inhomogeneous one

$$w = \sum_{k=1}^{4} A_k^0 w_k^0 + \sum_{n=1}^{\infty} \tilde{C}_n \psi_n + w_0. \tag{6}$$

Here $w_k^0 = \{\sinh p_1 x,\ \cosh p_1 x,\ \sin p_2 x,\ \cos p_2 x\}$, $p_{1,2}^2 = \pm p/2 + \sqrt{p^2/4 + q}$; $A_k^0$, $\tilde{C}_n$ and $w_0$ unknown constants.

We will represent $\tilde{C}_n$ and $w_0$ through unknown constants $A_k^0$. To do this, we substitute (6) into Eq. (3) in the first relation (4)

$$\tilde{C}_n = \omega^2 \frac{a_n}{k_n d_n - \omega^2 a_n} \sum_{k=1}^{4} A_k^0 E_{kn}^0, \qquad w_0 = -\sum_{k=1}^{4} \tilde{w}_k^0 A_k^0, \tag{7}$$

where

$$\begin{aligned}
& d_n = (Dk_n^2 + T)k_n^2 - g\rho - k_0\omega^2, \\
& \tilde{w}_k^0 = \frac{1}{2a}\int_{-a}^{a} w_k^0 dx = \Big\{0, \frac{\sinh \tilde{p}_1}{p_1}, \frac{\sin \tilde{p}_2}{p_2}\Big\}, \\
& E_{kn}^0 = \frac{1}{a}\int_{-a}^{a} w_k^0 \psi_n dx, \\
& E_{1n}^0 = \frac{p_1 \cosh \tilde{p}_1}{a(k_n^2 + p_1^2)}[(-1)^n - 1], \\
& E_{2n}^0 = \frac{p_1 \sinh \tilde{p}_1}{a(k_n^2 + p_1^2)}[(-1)^n + 1], \\
& E_{3n}^0 = \frac{p_2 \cos \tilde{p}_2}{a(k_n^2 - p_2^2)}[(-1)^n - 1], \\
& E_{4n}^0 = -\frac{p_2 \sin \tilde{p}_2}{a(k_n^2 - p_2^2)}[(-1)^n + 1], \\
& \tilde{p}_i = ap_i.
\end{aligned} \tag{8}$$

Taking into account relation (7), the final expression for the shape of the flexure of the plate (6) will take the form

$$w = \sum_{k=1}^{4} \Big(w_k^0 - \tilde{w}_k^0 - \omega^2 \sum_{n=1}^{\infty} \frac{a_n E_{kn}^0}{\omega^2 \tilde{a}_n - k_n \tilde{d}_n} \psi_n Brig) A_k^0. \tag{9}$$

Here $\tilde{a}_n = a_n + k_n k_0$, $\tilde{d}_n = (Dk_n^2 + T)k_n^2 - g\rho$.

From the boundary conditions for plate fixing (5), there are four linear homogeneous equations with respect to unknowns $A_k^0$

$$\begin{aligned}
& \sum_{k=1}^{4} \Big(B_{1k} - \omega^2 \sum_{n=1}^{\infty} \alpha_n E_{kn}^0 B_{1n}^*\Big) A_k^0 = 0, \qquad \sum_{k=1}^{4} C_{1k}^0 A_k^0 = 0, \\
& \sum_{k=1}^{4} \Big(B_{2k} - \omega^2 \sum_{n=1}^{\infty} \alpha_n E_{kn}^0 B_{2n}^*\Big) A_k^0 = 0, \qquad \sum_{k=1}^{4} C_{2k}^0 A_k^0 = 0,
\end{aligned} \tag{10}$$

where

$$B_{jk} = (w_k^0 - \tilde{w}_k^0)\Big|_{x=\pm a}, \quad B_{jn}^* = \begin{cases} 1, & j=1,\ (x=-a) \\ (-1)^n, & j=2,\ (x=a) \end{cases}, \quad C_{jk}^0 = \frac{dw_k^0}{dx}\Big|_{x=\pm a},$$

$$\alpha_n = a_n/(\omega^2 a_n - k_n d_n) = a_n/(\omega^2 \tilde{a}_n - k_n \tilde{d}_n).$$

The frequency equation of the natural oscillations of the elastic plate and the liquid is obtained from the equality to zero of the determinant of the homogeneous system (10)

$$\left| \left| C_{qk} \right|_{q,k=1}^{4} \right| = 0. \tag{11}$$

Here

$$\begin{aligned}
C_{11} &= \sinh \tilde{p}_1 - \omega^2 \sum_{m=1}^{\infty} \alpha_{2m-1} E_{1,2m-1}^0, \\
C_{12} &= \cosh \tilde{p}_1 - \frac{\sinh \tilde{p}_1}{p_1} - \omega^2 \sum_{m=1}^{\infty} \alpha_{2m} E_{2,2m}^0, \\
C_{13} &= \sin \tilde{p}_2 - \omega^2 \sum_{m=1}^{\infty} \alpha_{2m-1} E_{3,2m-1}^0, \\
C_{14} &= \cos \tilde{p}_2 - \frac{\sin \tilde{p}_2}{p_2} - \omega^2 \sum_{m=1}^{\infty} \alpha_{2m} E_{4,2m}^0. \\
& C_{21} = p_1 \cosh \tilde{p}_1, \quad C_{22} = -p_1 \sinh \tilde{p}_1, \quad C_{23} = p_2 \cos \tilde{p}_2, \\
& C_{24} = -p_2 \sin \tilde{p}_2, \quad C_{31} = -C_{11}, \quad C_{32} = C_{12}, \quad C_{33} = -C_{13}, \\
& C_{34} = C_{14}, \quad C_{41} = C_{21}, \quad C_{42} = -C_{22}, \quad C_{43} = C_{23}, \quad C_{44} = -C_{24}.
\end{aligned} \tag{12}$$

Using the expansion of the functions $w_k^0$ in series with respect to complete and orthogonal system of functions $\psi_n$, the coefficients $C_{11}$, $C_{12}$, $C_{13}$ and $C_{14}$ can be rewritten as

$$\begin{aligned}
C_{11} &= \sum_{m=1}^{\infty} \beta_{2m-1} E_{1,2m-1}^0, \quad C_{12} = \sum_{m=1}^{\infty} \beta_{2m} E_{2,2m}^0, \\
C_{13} &= \sum_{m=1}^{\infty} \beta_{2m-1} E_{3,2m-1}^0, \quad C_{14} = \sum_{m=1}^{\infty} \beta_{2m} E_{4,2m}^0,
\end{aligned} \tag{13}$$

where $\beta_n = 1 - \omega^2 \alpha_n = -k_n d_n / \left( \omega^2 \tilde{a}_n - k_n \tilde{d}_n \right)$.

It follows from the expression $E^0_{kn}$ the relations (8) that Eq. (11) can be written in the form

$$(14)\quad \Big(p_2\cos\tilde p_2\sum_{m=1}^{\infty}\beta_{2m-1}E^0_{1,2m-1}-p_1\cosh\tilde p_1\sum_{m=1}^{\infty}\beta_{2m-1}E^0_{3,2m-1}\Big)\times\Big(-p_1\sinh\tilde p_1\sum_{m=1}^{\infty}\beta_{2m}E^0_{4,2m}-p_2\sin\tilde p_2\sum_{m=1}^{\infty}\beta_{2m}E^0_{2,2m}\Big)=0.$$

Taking into account relations (8) and the equality

$$(k_n^2+p_1^2)(k_n^2-p_2^2)=\frac{1}{D}d_n\,,$$

the expressions

$$p_2\cos\tilde p_2 E^0_{1,2m-1}-p_1\cosh\tilde p_1 E^0_{3,2m-1}\quad\text{and}\quad -p_1\sinh\tilde p_1 E^0_{4,2m}-p_2\sin\tilde p_2 E^0_{2,2m}$$

take the form

$$p_2\cos\tilde p_2 E^0_{1,2m-1}-p_1\cosh\tilde p_1 E^0_{3,2m-1}=-\frac{2}{a}p_1p_2\cosh\tilde p_1\cos\tilde p_2\frac{p_2^2-p_1^2}{d_{2m-1}}D,$$

$$(15)\quad -p_1\sinh\tilde p_1 E^0_{4,2m}-p_2\sin\tilde p_2 E^0_{2,2m}=\frac{2}{a}p_1p_2\sinh\tilde p_1\sin\tilde p_2\frac{p_1^2-p_2^2}{d_{2m}}D.$$

$$-p_1\sinh\tilde p_1 E^0_{4,2m}-p_2\sin\tilde p_2 E^0_{2,2m}=\frac{2}{a}p_1p_2\sinh\tilde p_1\sin\tilde p_2\frac{p_1^2-p_2^2}{d_{2m}}D.$$

Substituting expressions (15) into Eq. (14), we obtain a simplified frequency equation:

$$(16)\quad \sum_{n=1}^{\infty}\frac{k_n}{\omega^2\tilde a_n-k_n\tilde d_n}=0.$$

Thus, the frequency Eq. (14) splits into two equations describing odd ($n=2m-1$) and even ($n=2m$) frequencies and can be written in a single form for these frequencies (16). It should be noted that this simplification was achieved due to expansion of a function $w^0_k$ in a series with respect to the complete and orthogonal system of Eigen-functions $\psi_n(x)$. Normal modes of free oscillations of the plate will be found from relations (9) and (10).

The form of equation (16) does not change when the plate degenerates into a membrane ($D=0$, $T>0$ and $\tilde d_n=Tk_n^2-g\rho$). In this case, Eq. (16) is the same as the equation of work [1].

The left-hand part of Eq. (16) is a monotonically increasing function of the parameter $\omega^2$ on the intervals $\{k_n\tilde{d}_n/\tilde{a}_n,\ k_{n+1}\tilde{d}_{n+1}/\tilde{a}_{n+1}\}$ $(n = 1, 2, \ldots)$, the function takes on its values from $-\infty$ to $\infty$. Therefore, between two successive values $k_n\tilde{d}_n/\tilde{a}_n$ there only one root of Eq. (16). This determines in advance the intervals, in which the natural frequencies are located. On the increase of number of terms of the series, the previous roots will be refined and new ones will appear. Squares of high frequencies ($n \gg 1$) will be little different from the magnitude

$$\omega_n^2 = k_n \frac{(Dk_n^2 + T)k_n^2 - g\rho}{\rho \coth \kappa_n + k_n k_0}\,.$$

It follows from this equality that $Dk_n^2 + T > g\rho/k_n^2$, the addiction to $\omega_n^2$ depending on $D$ and $T$ is linear, the greatest value $\omega_n^2$ will be when $k_0 = 0$, i.e. for the inertia-free plate. Square of frequencies $\omega_n^2$ are slightly dependent on the filling depth $h$ and decrease with its reduction. Thus, the problem under consideration has an infinite discrete spectrum of eigenvalues $\omega_l^2$, which are the roots of Eq. (16), and the corresponding eigenfunctions $w_l(x)$ form a complete orthogonal system of functions on the segment $[-a,\ a]$.

However, the problem under consideration has a number of physical features. In order that there is no rupture of continuity (formation of cavitation), the pressure inside the liquid must be non-negative. For this, the external pressure on the lower elastic foundation $P_0$ should not be less than the value $g(\rho h + k_0)$. This inequality does not take into account the magnitude of the plate bending stiffness $D$ and tension $T$. Critical values of the bending stiffness and the tension will be found from the stability conditions of the vibrations of the plate and liquid.

## 4 Approximate and Accurate Conditions of Stability of Coupled Oscillations of the Plate and Liquid. Dynamic Approach

If we retain two terms in the series of equation (16), then we obtain an approximate frequency equation, whose solution in dimensionless variables for odd ($n = 1, 3$) and even ($n = 2, 4$) frequencies, respectively, takes the form

$$\Omega^2 = \frac{2(5\pi^2\gamma - \tilde{g} + 41\pi^4)}{3\coth\pi\tilde{h} + \coth 3\pi\tilde{h} + 6\pi\tilde{k}_0}\pi,$$
$$\Omega^2 = \frac{2(10\pi^2\gamma - \tilde{g} + 136\pi^4)}{2\coth 2\pi\tilde{h} + \coth 4\pi\tilde{h} + 4\pi\tilde{k}_0}\pi. \qquad (17)$$

Here $\Omega^2 = \omega^2\rho b^3/D$, $\gamma = Tb^2/D$, $\tilde{h} = h/b$, $\tilde{k}_0 = k_0/\rho b$, $\tilde{g} = \rho g b^4/D$.

It follows from formulas (16) that the dependence of the square of a dimensionless frequency on a dimensionless $\gamma$ is linear, the inertia-free plate has the greatest frequency, frequencies almost do not dependent on the filling depth at $\tilde{h} > 1$ and decrease with depth decreasing.

From the inequality $\Omega^2 > 0$ we obtain the approximate stability conditions for the oscillations of the plate and the liquid

$$\begin{aligned} &5\pi^2\gamma + 41\pi^4 > \tilde{g} \quad (n = 1, 3), \\ &10\pi^2\gamma + 136\pi^4 > \tilde{g} \quad (n = 2, 4). \end{aligned} \tag{18}$$

It follows from (18) that under compressive stresses ($T < 0$), stability conditions (18) worsen.

At $D = 0$ and $T > 0$ (the case of the membrane), formulas (17) – (18) take the form [1,4]:

$$\begin{aligned} \Omega^2 &= \frac{6\pi(5\pi^2\tilde{T} - 1)}{3\coth\pi\tilde{h} + \coth 3\pi\tilde{h} + 6\pi\tilde{k}_0} \quad (n = 1, 3), \\ \Omega^2 &= \frac{8\pi(10\pi^2\tilde{T} - 1)}{2\coth 2\pi\tilde{h} + \coth 4\pi\tilde{h} + 4\pi\tilde{k}_0} \quad (n = 2, 4). \end{aligned} \tag{19}$$

$$\begin{aligned} \tilde{T} &> \frac{1}{5\pi^2} \approx 0.025325 \quad (n = 1, 3), \\ \tilde{T} &> \frac{1}{10\pi^2} \approx 0.0123819 \quad (n = 2, 4). \end{aligned} \tag{20}$$

Here $\Omega^2 = \omega^2 b/g$, $\tilde{T} = T/g\rho b^2 > 0$.

To clarify conditions (18), we proceed in the same way as in [1,4]. In Eq. (16) we set $\omega^2 = 0$ and solve the resulting equation with respect to the critical values of the parameters. When $\omega^2 = 0$, Eq. (16) takes the form $\sum_{n=1}^{\infty} 1/\tilde{d}_n = 0$, or in dimensionless variables

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + \beta n^2 - \alpha^4} = 0, \tag{21}$$

where $\alpha^4 = \rho b^4/D\pi^4 > 0$, $\beta = Tb^2/D\pi^2$.

At $T = 0$, equation (21) can be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 - \alpha^4} = 0. \tag{22}$$

Number series $\sum_{n=1}^{\infty} 1/(n^4 - \alpha^4) = 0$ for odd ($n = 2m - 1$) and even ($n = 2m$) values $n$ can be represented as

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^4 - \alpha^4} = \pi \frac{\tan \pi\alpha/2 - \tanh \pi\alpha/2}{8\alpha^3},$$
(23)
$$\sum_{m=1}^{\infty} \frac{1}{(2m)^4 - \alpha^4} = -\pi \frac{\pi\alpha \cot \pi\alpha/2 - \pi\alpha \coth \pi\alpha/2 - 4}{8\alpha^4}.$$

Taking into account (23) the first root of Eq. (22) with $n = 2m - 1$ has the form $\pi\alpha/2 = 3.926602312$, from which the updated stability condition follows:

(24) $$D > 2.62913 \times 10^{-4} g\rho b^4.$$

Taking into account (23) the first root of Eq. (22) with $n = 2m$ has the form $\pi\alpha/2 = 5.2676575303$ from which the updated stability condition follows:

(25) $$D > 8.11725 \times 10^{-5} g\rho b^4.$$

Approximate conditions prescribed inequality (18) for $n = 1, 3$ and $n = 2, 4$ when $T = 0$, respectively, can be written as

(26) $$D > 2.50390 \times 10^{-4} g\rho b^4, \qquad D > 7.54852 \times 10^{-5} g\rho b^4.$$

It is seen from formulas (24) – (26) that the approximate value of critical bending stiffness for asymmetric frequencies is 0.952 times lower, and for symmetric frequencies it is 0.930 times lower. It should be noted that proximity of the approximate value and the exact takes place, taking into account two terms of the series we obtain sufficient accuracy for practice.

The numerical series $\sum_{n=1}^{\infty} 1/(n^4 + \beta n^2 - \alpha^4) = 0$ for $\beta \neq 0$ has a more complex representation, which complicates its further study.

## 5 Static Stability Conditions for the Equilibrium Position of the Plate and Liquid. Static Approach

In the case of a static analysis of the perturbation problem, we have

(27) $$\frac{d^4W}{dx^4} - p\frac{d^2W}{dx^2} - qW = \tilde{C},$$

(28) $$W\Big|_{x=\pm a} = 0\,, \quad \frac{dW}{dx}\Big|_{x=\pm a} = 0\,, \quad \int_{-a}^{a} W\,dx = 0,$$

where $p = T/D$, $q = g\rho/D$, $\tilde{C} = (P_0 + Q)/D$.

Equation (27) has a solution

(29) $$W = A_1 \cosh p_1 x + A_2 \sinh p_1 x + A_3 \cos p_2 x + A_4 \sin p_2 x + C.$$

Here $C = -P_0 + Q)/\rho g$, $p_{1,2}^2 = \pm p/2 + \sqrt{p^2/4 + q}$.

The constant $Q$ is unknown. Substituting (29) into the boundary conditions and the incompressibility condition for the liquid (28), we obtain a homogeneous system of equations with respect to unknown $A_k$ $(k = \overline{1,4})$ and $C$

$$
\begin{aligned}
&A_1 \cosh\tilde{p}_1 - A_2 \sinh\tilde{p}_1 + A_3 \cos\tilde{p}_2 - A_4 \sin\tilde{p}_2 + C = 0,\\
&-A_1 p_1 \sinh\tilde{p}_1 + A_2 p_1 \cosh\tilde{p}_1 + A_3 p_2 \sin\tilde{p}_2 + A_4 p_2 \cos\tilde{p}_2 = 0,\\
&A_1 \cosh\tilde{p}_1 + A_2 \sinh\tilde{p}_1 + A_3 \cos\tilde{p}_2 + A_4 \sin\tilde{p}_2 + C = 0,\\
&A_1 p_1 \sinh\tilde{p}_1 + A_2 p_1 \cosh\tilde{p}_1 - A_3 p_2 \sin\tilde{p}_2 + A_4 p_2 \cos\tilde{p}_2 = 0,\\
&A_1 \sinh\tilde{p}_1/p_1 + A_3 \sin\tilde{p}_2/p_2 + Ca = 0,
\end{aligned}
\tag{30}
$$

where $\tilde{p}_i = p_i a$.

For existence of nonzero solution of the homogeneous system (30) its determinant must be equal to zero

$$
(p_1 \tan\tilde{p}_2 - p_2 \tanh\tilde{p}_1)\left(p_1 \cot\tilde{p}_2 - p_2 \coth\tilde{p}_1 - \frac{p_1^2 + p_2^2}{a p_1 p_2}\right) = 0. \tag{31}
$$

Equation (31) splits into two equations

$$
\begin{aligned}
&p_1 \tan\tilde{p}_2 - p_2 \tanh\tilde{p}_1 = 0 \ \text{ and}\\
&p_1 \cot\tilde{p}_2 - p_2 \coth\tilde{p}_1 - \frac{p_1^2 + p_2^2}{a p_1 p_2} = 0.
\end{aligned}
\tag{32}
$$

The first equation describes asymmetric static deflections of the plate, and the second describes symmetric ones.

At $T = 0$ ($p_1 = p_2 = \sqrt{q} = \alpha\pi/b$) these equations coincide with the Eqs. (23).

It should be noted that the values of critical stiffness found according to the static approach coincide with the exact values obtained by the dynamic approach. Thus, the simple static approach makes it possible to determine exactly the stability conditions for coupled oscillations of a plate and a liquid.

## 6 Summary and Directions for Further Research

Based on the analytical and numerical studies, we can conclude the following:

1. Natural oscillation frequencies of the plate and liquid for asymmetric and symmetric frequencies are presented in a single form.

2. An approximate formula for high frequencies is obtained. The dependence of the square of the frequencies on the flexural rigidity and the tension is linear. The highest frequency value will occur for an inertia-free plate. Frequencies decrease with the liquid depth decreasing.

3. Taking into account two terms of the series, an approximate formula is obtained for a square of dimensionless frequency. The main conclusions from this formula coincide with the results obtained for high frequencies.

4. The approximate stability conditions for the coupled vibrations of the plate and liquid are derived. These conditions are independent of the depth of the liquid and the mass of the plate.

5. Exact stability conditions are obtained. There is a closeness between the approximate value and the exact one. The stability conditions of the static approach coincide with the exact stability conditions of the dynamic approach.

6. It is shown that the approximate value of the critical bending stiffness for asymmetric frequencies are 0.952 times lower, and for symmetric frequencies they are 0.930 times lower. Thus, taking into account two terms of the series gives accuracy sufficient for practice.

7. Using numerical calculations we establish that with the increase of the number of terms in series of the frequency equation the previous frequencies are refined and new ones appear. To obtain acceptable accuracy, it is enough to take into account 10-20 terms of the series.

In the future, it is planned to consider this problem, taking into account the elastic upper and lower bases and study the frequency spectrum and stability conditions.

## ACKNOWLEDGEMENTS

The study was carried out within the Fundamental Research Programme funded by the Ministry of Education and Science of Ukraine, Project No. 0119U100042.

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