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SOLUTION OF THE PROBLEM OF FREE VIBRATIONS OF A NONTHIN ORTHOTROPIC SHALLOW SHELL OF VARIABLE THICKNESS IN THE REFINED STATEMENT

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We consider the problem of investigation of the spectrum of natural vibrations of a nonthin orthotropic shallow shell variable in two coordinate directions of thickness in the nonclassical statement. The approach to the solution of the obtained two-dimensional boundary-value problem is based on its reduction (by the method of spline-approximation of the unknown functions along one coordinate direction) to the one-dimensional problem with its subsequent solution. We study different cases of boundary conditions imposed on the contours of the shell. We also perform the comparison and analysis of the natural frequencies and modes of vibrations of orthotropic shells of constant and variable thickness.

Introduction

Shallow anisotropic shells are extensively used in various branches of contemporary engineering, aircraft building, shipbuilding, rocket production, etc. In the industry and civil engineering, numerous structural components have the form of shells with different geometric and physical parameters. In the course of development and implementation of new technologies, the requirements to the strength parameters and the reliability of created machines, mechanisms, and structures, including those with the shape of shells become more and more severe. Hence, it is necessary to develop the efficient numerical and experimental methods for the investigation objects of this kind with an aim to determine the parameters of their load-carrying ability and, in particular, the resonance frequencies of vibration of the shells.

The scientific works devoted to the investigation of problems of this kind mainly use the classical theory based on the Kirchhoff–Love hypotheses [1–4, 6]. The fundamentals of the classical theory of shells can be found in [1, 11]. In the numerical analysis of some types of the shells, it is necessary to apply the refined versions of the theory connected with the use of certain additional parameters. In particular, in the case of nonthin shells, it is reasonable to take into account transverse shears, which is done in the Timoshenko–Mindlin nonclassical theory [12–14]. In fact, we can mention numerous monographs, where the reader can find the presentation of the fundamentals of the refined theory of shells. The results of investigations of the stress-strain state of shallow shells in the nonclassical statement are presented in [8, 9]. The numerical analyses of the free vibrations of plates in the Timoshenko–Mindlin theory can be found in [5, 8, 13, 15].

In the present paper, we consider free vibrations of an orthotropic shallow cylindrical shell of rectangular shape (in plan) for different types of boundary conditions. The thickness of the shell varies in both coordinate directions. The analyses of frequencies are based on the use of spline approximation along one of coordinate

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directions and subsequent solution of the boundary-value problem for eigenvalues for systems of ordinary differential equations of high order with variable coefficients by the stable numerical method of discrete orthogonalization in combination with the method of step-by-step search [7–10].

The aim of investigations carried out in the present paper is to find the frequencies of free vibrations of elastic rectangular shells whose thickness vary in both coordinate directions in the refined statement on the basis of the method of spline approximation and compare the values of frequencies obtained for different types of boundary conditions on the contours of the shell.

Basic Relations

Consider the problem of free vibrations of a shallow orthotropic shell of rectangular shape (in plan)

$$0 \le x \le a, \quad 0 \le y \le b, \quad -\frac{h}{2} \le z \le \frac{h}{2},$$

whose thickness $h(x,y)$ varies in two coordinate directions. The geometry of the shell (in plan) is approximately identified with the geometry of the middle surface of the shell and the principal curvatures satisfy the relation

$$k_1 \cdot k_2 \approx 0 .$$

The reasoning is performed with the use of the Timoshenko–Mindlin hypothesis according to which an element normal to the coordinate surface prior to deformation preserves its length and remains rectilinear after deformation but is no longer perpendicular to the surface and rotates by a certain angle. It is also taken into account that the normal stresses in the planes parallel to the coordinate surface are small as compared to the corresponding stresses on the surfaces perpendicular to this coordinate surface [8].

According to the accepted hypothesis, we represent the displacements u_x, u_y, and u_z in the form

$$\begin{aligned} u_x(x,v,z,t) &= u(x,y,t) + z\psi_x(x,y,t), \\ u_y(x,v,z,t) &= v(x,y,t) + z\psi_y(x,y,t), \\ u_z(x,v,z,t) &= w(x,y,t), \end{aligned} \tag{1}$$

where x, y, and z are the coordinates of points of the shell, t is time, u_x, u_y, and u_z are the corresponding displacements; u, v, w are the displacements of points of the coordinate surface in the directions x, y, and z, respectively, and ψ_x and ψ_y are the total angles of rotation of the rectilinear element.

According to (1), we write expressions for the strains in the form

$$\begin{aligned} e_x(x,y,z,t) &= \varepsilon_x(x,y,t) + z\chi_x(x,y,t), \\ e_y(x,y,z,t) &= \varepsilon_y(x,y,t) + z\chi_y(x,y,t), \\ e_{xy}(x,y,z,t) &= \varepsilon_{xy}(x,y,t) + z2\chi_{xy}(x,y,t), \end{aligned}$$

$$e_{xz}(x,y,z,t) \cong \gamma_x(x,y,t),$$

$$e_{yz}(x,y,z,t) \cong \gamma_y(x,y,t).$$

Here, γ_x and γ_y are the angles of rotation caused by transverse shears, ε_x, ε_y, and ε_{xy} are the components of tangential strains specifying the internal geometry of the coordinate surface, and χ_x, χ_y, and $2\chi_{xy}$ are the components of bending strains characterizing the processes of bending and torsion of the coordinate surface.

The equations used to describe free transverse vibrations of the shallow shells within the framework of the Timoshenko–Mindlin-type nonclassical theory take the form [8]

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0,$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} - k_1 N_x - k_2 N_y + \rho h \omega^2 w = 0, \tag{2}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x + \rho \frac{h^3}{12} \omega^2 \psi_x = 0,$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y + \rho \frac{h^3}{12} \omega^2 \psi_y = 0.$$

In Eqs. (2), x and y are the rectangular Cartesian coordinates of the middle surface $0 \le x \le a$, $0 \le y \le b$, w is a deflection, and ρ is the density of the material.

The relations of elasticity are true for the normal N_x, N_y and shear N_{xy}, N_{yx} forces, bending M_x, M_y and torque M_{xy}, M_{yx} moments, and the lateral forces Q_x, Q_y. In the case of an orthotropic shell for which the axes of orthotropy coincide with coordinate axes, these relations can be represented as follows:

$$N_x = C_{11}\left(\frac{\partial u}{\partial x} + k_1 w\right) + C_{12}\left(\frac{\partial v}{\partial y} + k_2 w\right),$$

$$N_y = C_{12}\left(\frac{\partial u}{\partial x} + k_1 w\right) + C_{22}\left(\frac{\partial v}{\partial y} + k_2 w\right),$$

$$N_{xy} = C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + k_2 D_{66}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right),$$

$$N_{yx} = C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + k_1 D_{66}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right),$$

$$M_x = D_{11}\left(\frac{\partial\psi_x}{\partial x} + k_1^2 w\right) + D_{12}\left(\frac{\partial\psi_y}{\partial y} + k_2^2 w\right),$$

$$M_y = D_{12}\left(\frac{\partial\psi_x}{\partial x} + k_1^2 w\right) + D_{22}\left(\frac{\partial\psi_y}{\partial y} + k_2^2 w\right),$$

$$M_{xy} = M_{yx} = D_{66}\left(\frac{\partial\psi_x}{\partial y} + \frac{\partial\psi_y}{\partial x}\right),$$

$$Q_x = K_1\gamma_x = K_1\left(\psi_x + \frac{\partial w}{\partial x} - k_1 u\right),$$

$$Q_y = K_2\gamma_y = K_2\left(\psi_y + \frac{\partial w}{\partial y} - k_2 v\right).$$

We now determine the stiffness characteristics C_{ij}, K_i, and D_{ij} by the following formulas:

$$C_{11} = \frac{E_x h}{1-\nu_x\nu_y}, \quad C_{12} = \nu_y C_{11}, \quad C_{22} = \frac{E_y h}{1-\nu_x\nu_y}, \quad C_{66} = G_{xy}h,$$

$$D_{11} = \frac{E_x h^3}{12(1-\nu_x\nu_y)}, \quad D_{12} = \nu_y D_{11},$$

$$D_{22} = \frac{E_y h^3}{12(1-\nu_x\nu_y)}, \quad D_{66} = \frac{G_{xy}h^3}{12}, \tag{3}$$

$$K_1 = \frac{5}{6}hG_{xz}, \quad K_2 = \frac{5}{6}hG_{yz}.$$

In Eqs. (3), E_x and E_y are the moduli of elasticity, G_{xy}, G_{xz}, and G_{yz} are the shear moduli, and ν_x and ν_y are Poisson's ratios.

On the contours of the shell

$$x = 0, \quad x = a \quad \text{and} \quad y = 0, \quad y = b,$$

we impose the boundary conditions represented in terms of displacements and angles of rotation.

We represent expressions for the boundary conditions and $x = \text{const}$:

— in the case of a rigidly fixed contour

$$u = v = w = \psi_x = \psi_y = 0 \quad \text{for} \quad x = 0, \quad x = a; \tag{4}$$

— in the case of a hinged contour

$$\frac{\partial u}{\partial x} = v = w = \frac{\partial \psi_x}{\partial x} = \psi_y = 0 \quad \text{for} \quad x = 0, \quad x = a; \tag{5}$$

— one contour is rigidly fixed and the other contour is hinged

$$\begin{gathered} u = v = w = \psi_x = \psi_y = 0 \quad \text{for} \quad x = 0, \\ \frac{\partial u}{\partial x} = v = w = \frac{\partial \psi_x}{\partial x} = \psi_y = 0 \quad \text{for} \quad x = a. \end{gathered} \tag{6}$$

We can write similar boundary conditions on the contours $y = 0$ and $y = b$ by the change of variables

$$x \to y, \quad u \to v, \quad \text{and} \quad \psi_x \to \psi_y$$

in Eqs. (4)–(6).

To solve the obtained two-dimensional boundary-value problem, we use an approach based on the approximation of desired functions in one coordinate direction with the help of spline functions and solve the one-dimensional boundary-value problem thus obtained by the stable numerical method of discrete orthogonalization in combination with the method of step-by-step search [7–10].

Procedure of Solution

We seek the solution of the system of equations (2) in the form

$$\begin{gathered} u(x,y) = \sum_{i=0}^{N} u_i(x)\varphi_{1,i}(y), \quad v(x,y) = \sum_{i=0}^{N} v_i(x)\varphi_{2,i}(y), \\ w(x,y) = \sum_{i=0}^{N} w_i(x)\varphi_{3,i}(y), \\ \psi_x(x,y) = \sum_{i=0}^{N} \psi_{x_i}(x)\varphi_{4,i}(y), \quad \psi_y(x,y) = \sum_{i=0}^{N} \psi_{y_i}(x)\varphi_{5,i}(y), \end{gathered} \tag{7}$$

where

$$u_i(x), \quad v_i(x), \quad w_i(x), \quad \psi_{x_i}(x), \quad \text{and} \quad \psi_{y_i}(x), \quad i = 0,\ldots,N,$$

are the desired functions, $\varphi_{j,i}(y)$, $j = 1,\ldots,5$, are linear combinations of B-splines on the uniform mesh Δ: $0 = y_0 < y_1 < \ldots < y_N = b$, satisfying the boundary conditions on the contours $y = 0$ and $y = b$.

In the analyzed case, we restrict ourselves to the approximation by spline functions of the third degree:

$$B_3^i(y) = \frac{1}{6}\begin{cases} 0, & -\infty < y < y_{i-2}, \\ z^3, & y_{i-2} \le y < y_{i-1}, \\ -3z^3 + 3z^2 + 3z + 1, & y_{i-1} \le y < y_i, \\ 3z^3 - 6z^2 + 4, & y_i \le y < y_{i+1}, \\ (1-z)^3, & y_{i+1} \le y < y_{i+2}, \\ 0, & y_{i+2} \le y < \infty, \end{cases}$$

where $z = (y - y_k)/h_y$ in the interval $[y_k, y_{k+1}]$, $k = i-2, \ldots, i+1$, $i = -1, \ldots, N+1$, $h_y = y_{k+1} - y_k = \text{const}$.

In this case, we construct the functions $\varphi_{j,i}(y)$ as follows:

— if the corresponding resolving function is equal to zero, then

$$\varphi_{j,0}(y) = -4B_3^{-1}(y) + B_3^0(y), \qquad \varphi_{j,1}(y) = B_3^{-1}(y) - \frac{1}{2}B_3^0(y) + B_3^1(y),$$

$$\varphi_{j,i}(y) = B_3^i(y), \quad i = 2, 3, \ldots, N-2;$$

— if the derivative of the resolving function with respect to y is equal to zero, then

$$\varphi_{j,0}(y) = B_3^0(y), \qquad \varphi_{j,1}(y) = B_3^{-1}(y) - \frac{1}{2}B_3^0(y) + B_3^1(y),$$

$$\varphi_{j,i}(y) = B_3^i(y), \quad i = 2, 3, \ldots, N-2.$$

We also have similar formulas for the functions $\varphi_{j,N-1}(y)$ and $\varphi_{j,N}(y)$.

Substituting (7) in the system of equations (2), we require that Eqs. (2) hold at given collocation points $\xi_k \in [0, b]$, $k = 0, \ldots, N$. In the case of the even number of nodes of the mesh $(N = 2n+1,\ n \ge 3)$ and under the condition that $\xi_{2i} \in [y_{2i}, y_{2i+1}]$, $\xi_{2i+1} \in [y_{2i}, y_{2i+1}]$, $i = 0, \ldots, n$, on the segment $[y_{2i}, y_{2i+1}]$, we get two collocation nodes, while on the neighboring segments $[y_{2i+1}, y_{2i+2}]$, the collocation nodes are absent. On each segment $[y_{2i}, y_{2i+1}]$, we choose collocation points as follows:

$$\xi_{2i} = y_{2i} + z_1 h, \qquad \xi_{2i+1} = y_{2i} + z_2 h, \qquad i = 0, \ldots, n,$$

where z_1, z_2 are the roots of the second-order Legendre polynomial on the segment $[0,1]$:

$$z_{1,2} = \frac{1}{2} \mp \frac{\sqrt{3}}{6}.$$

This choice of collocation points is optimal and noticeably increases the order of accuracy of the approximation. After transformations, we obtain a system of $N+1$ linear differential equations for u_i, v_i, w_i, ψ_{x_i}, and ψ_{y_i}.

The obtained system of ordinary differential equations can be reduced to the normal form as follows:

$$\frac{d\bar{Y}}{dx} = A(x,\omega)\bar{Y}, \quad 0 \le x \le a, \tag{8}$$

where

$$\bar{Y} = \left[\bar{u}, \bar{u}', \bar{v}, \bar{v}', \bar{w}, \bar{w}', \bar{\psi}_x, \bar{\psi}'_x, \bar{\psi}_y, \bar{\psi}'_y\right]^\top$$

$$= \left[u_0, \ldots, u'_N, v_0, \ldots, v'_N, w_0, \ldots, w'_N, \psi_{x_0}, \ldots, \psi'_{x_N}, \psi_{y_0}, \ldots, \psi'_{y_N}\right]^\top$$

is the column vector of the desired functions and their derivatives of dimension $10(N+1)$ and $A(x,\omega)$ is a square matrix of order $10(N+1)\times 10(N+1)$.

In a similar way, we formulate the boundary conditions (4)–(6) for the system of equations (2)

$$B_1\bar{Y}(0) = \bar{0}, \qquad B_2\bar{Y}(a) = \bar{0}. \tag{9}$$

The problem of eigenvalues for the system of ordinary differential equations (8) with the boundary conditions (9) was solved by the method of discrete orthogonalization in combination with the method of step-by-step search [8, 9].

Results of Investigation

By using the proposed methods, we studied the spectrum of natural frequencies of vibrations of an orthotropic shallow shell of thickness $h(x,y)$ variable in two coordinate directions. We consider a square (in plan) cylindrical shell with $a = b = 0.5$ m and a dimensionless radius of curvature

$$r_x = \frac{R_x}{a} = 1.3$$

such that $k_x = 1.538\ \text{m}^{-1}$, and $k_y = 0$.

The thickness of the shell varies according to the law

$$h(x,y) = h_0\left(1+\alpha\cos\frac{\pi x}{a}\right)\left(1+\beta\cos\frac{\pi y}{b}\right), \tag{10}$$

where the parameters $|\alpha| \le 0.5$ and $|\beta| \le 0.5$ vary with steps of 0.1 and h_0 is the thickness of the shell of constant thickness and equivalent mass (we set $h_0 = 0.04$ m).

The physical parameters of the material of the shell are as follows:

$$E_x = 3.68 \cdot 10^{10}\ \text{Pa}, \quad E_y = 2.68 \cdot 10^{10}\ \text{Pa}, \quad G_{xy} = 0.50 \cdot 10^{10}\ \text{Pa},$$

$$G_{yz} = 0.41 \cdot 10^{10}\ \text{Pa}, \quad G_{xz} = 0.45 \cdot 10^{10}\ \text{Pa}, \quad \nu_x = 0.077, \quad \nu_y = 0.105, \quad \rho = 1870\ \text{kg/m}^3.$$

The posed problem was solved for five types of boundary conditions on the contours of the shell:

(1°) hinged support of all sides of the shell (boundary conditions of the ***BC-1°*** type):

$$\frac{\partial u}{\partial x} = v = w = \frac{\partial \psi_x}{\partial x} = \psi_y = 0 \quad \text{for} \quad x = 0, \quad x = a,$$

$$u = \frac{\partial v}{\partial y} = w = \psi_x = \frac{\partial \psi_y}{\partial y} = 0 \quad \text{for} \quad y = 0, \quad y = b;$$

(2°) rigid fixing along the entire contour (***BC-2°***):

$$u = v = w = \psi_x = \psi_y = 0 \quad \text{for} \quad x = 0, \quad x = a,$$

$$u = v = w = \psi_x = \psi_y = 0 \quad \text{for} \quad y = 0, \quad y = b;$$

(3°) rigid fixing of three sides and hinged support of the fourth side (***BC-3°***):

$$u = v = w = \psi_x = \psi_y = 0 \quad \text{for} \quad x = 0, \quad x = a, \quad y = 0,$$

$$u = \frac{\partial v}{\partial y} = w = \psi_x = \frac{\partial \psi_y}{\partial y} = 0 \quad \text{for} \quad y = b;$$

(4°) rigid fixing of two opposite sides and hinged support of the other two sides (***BC-4°***):

$$u = v = w = \psi_x = \psi_y = 0 \quad \text{for} \quad x = 0, \quad x = a,$$

$$u = \frac{\partial v}{\partial y} = w = \psi_x = \frac{\partial \psi_y}{\partial y} = 0 \quad \text{for} \quad y = 0, \quad y = b;$$

(5°) rigid fixing of two adjacent sides and hinged support of the other two sides (***BC-5°***):

$$u = v = w = \psi_x = \psi_y = 0 \quad \text{for} \quad x = 0, \quad y = 0,$$

Table 1

	$\bar{\omega}_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$		
i	***I***	***II***	Π, %
1	18.0714	18.0723	0.01
2	39.1647	39.1656	0.00
3	40.3978	40.4562	0.14
4	55.0759	55.1162	0.07

$$\frac{\partial u}{\partial x} = v = w = \frac{\partial \psi_x}{\partial x} = \psi_y = 0 \quad \text{for} \quad x = a,$$

$$u = \frac{\partial v}{\partial y} = w = \psi_x = \frac{\partial \psi_y}{\partial y} = 0 \quad \text{for} \quad y = b.$$

The accuracy of evaluation of the frequencies of the free vibrations in the investigated shell by the spline-collocation method was checked by comparing the obtained results with the corresponding values of frequencies computed analytically by expanding unknown functions in double Fourier series by the formulas

$$u = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} a_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

$$v = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} b_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b},$$

$$w = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} c_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \tag{11}$$

$$\psi_x = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} d_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b},$$

$$\psi_y = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} e_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}.$$

In Table 1, we present the values of the first four dimensionless frequencies of vibrations $\bar{\omega}_i$ of the shell of constant thickness $(h(x,y) = h_0,\ \alpha = 0,\ \beta = 0)$. Thus, in column ***I***, we give the frequencies analytically computed by using relation (11). The corresponding frequencies obtained as a result of calculations according to

Table 2

	$\bar{\omega}_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$ (for $\alpha = -0.4$, $\beta = -0.4$, and ***BC-2°***)							
i \ N	8	10	12	14	16	18	20	22
1	37.8216	37.6029	37.5358	37.5052	37.4896	37.4775	37.4736	37.4707
2	50.1194	49.6381	49.4826	49.4184	49.3815	49.3601	49.3460	49.3378
3	57.2793	55.1016	54.4211	54.1509	54.0187	53.9438	53.8996	53.8728
4	70.5151	68.6243	67.9049	67.6240	67.4874	67.4081	67.3610	67.3304

Table 3

	$\bar{\omega}_i = \omega_i a^2 \sqrt{\rho h / D_{11}}$ (for $\alpha = -0.4$, $\beta = 0.4$, and ***BC-5°***)							
i \ N	8	10	12	14	16	18	20	22
1	24.2309	24.0705	24.0156	23.9884	23.9738	23.9660	23.9607	23.9573
2	41.9581	41.3797	41.0915	40.9476	40.8703	40.8241	40.7954	40.7775
3	45.8808	44.6802	44.2145	44.0298	43.9408	43.8947	43.8660	43.8490
4	61.4790	60.6186	60.0164	59.7364	59.5949	59.5157	59.4676	59.4370

the spline-approximation method in the case of hinged fixing of all contours of the shell (***BC-1°***) for the number of collocation points $N = 18$ are presented in column ***II*** and Π is the difference expressed in terms of percent. As we can see, the difference between the frequencies does not exceed 0.2 %, which means that the accuracy and reliability of the method proposed for the evaluation of the free frequencies of vibrations of the shells are quite high.

We studied the dependence of the values of frequencies $\bar{\omega}_i$ computed by the proposed spline-collocation method on the number of collocation points N. The results of investigations are presented in Tables 2 and 3. The number of collocation points changes from $N = 8$ to $N = 22$ with steps equal to 2.

In Table 2, we present the first four dimensionless frequencies of vibrations of a shell of variable thickness for $\alpha = -0.4$, $\beta = -0.4$, and the ***BC-2°***-type boundary conditions on the contours of the shell and for $\alpha = 0.4$, $\beta = 0.4$. In Table 3, we give the corresponding data for the ***BC-5°***-type boundary conditions.

As follows from Tables 2 and 3, the frequencies of vibrations strongly depend on the number of collocation points for small values of N any any character of changes in the thickness of the shell and any boundary conditions on the edges. As the number of collocation points increases, the frequencies decrease but the accuracy of calculations increases. Even for $N = 16$, in most cases, it is possible to get a satisfactory accuracy of the results. In our investigation, all calculations were carried out for $N = 18$.

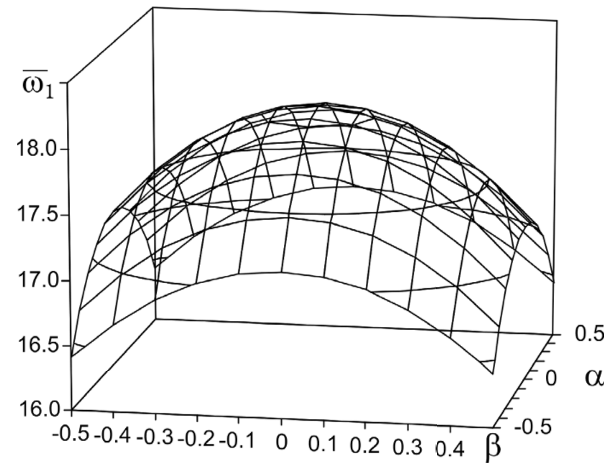

Fig. 1

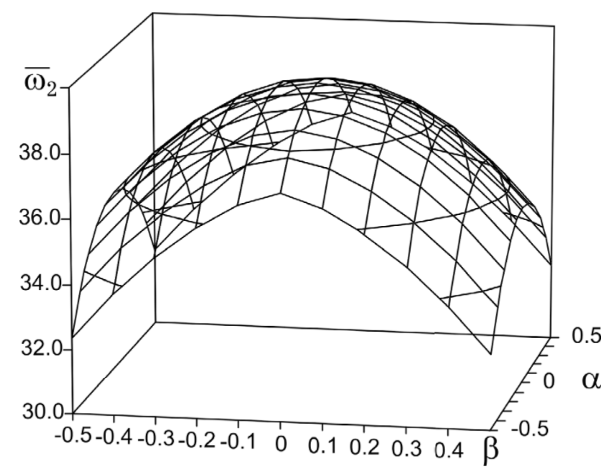

Fig. 2

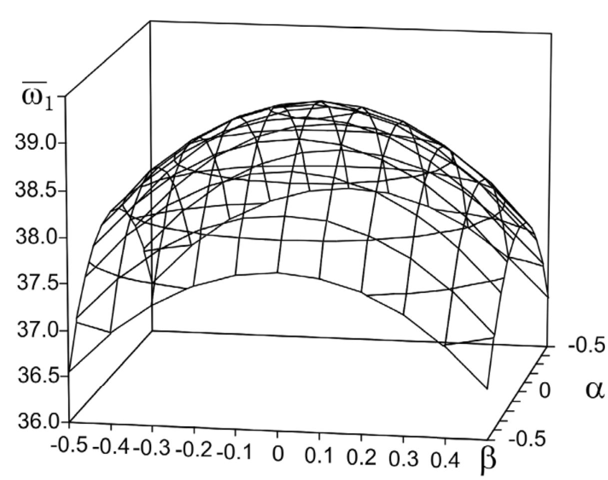

Fig. 3

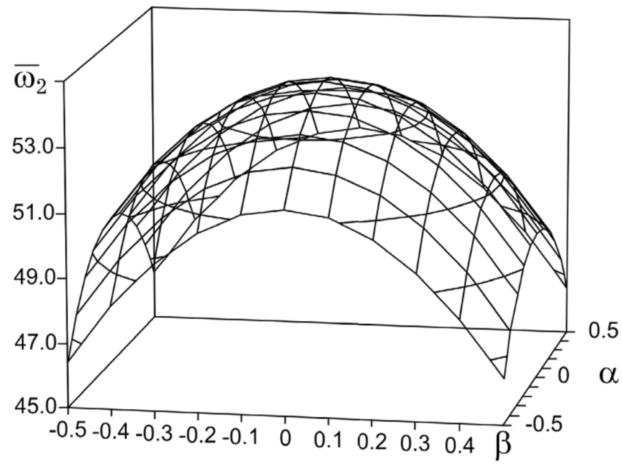

Fig. 4

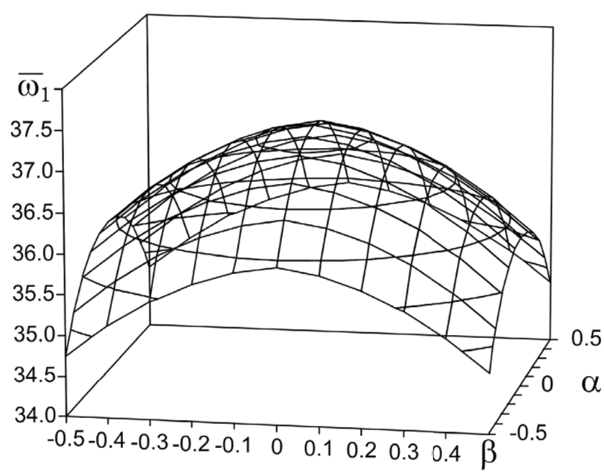

Fig. 5

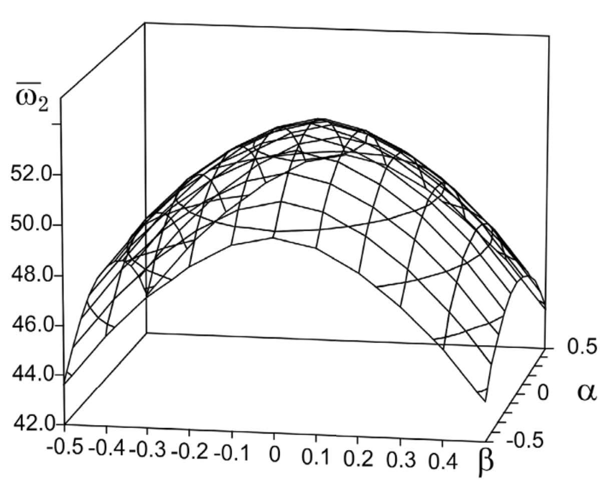

Fig. 6

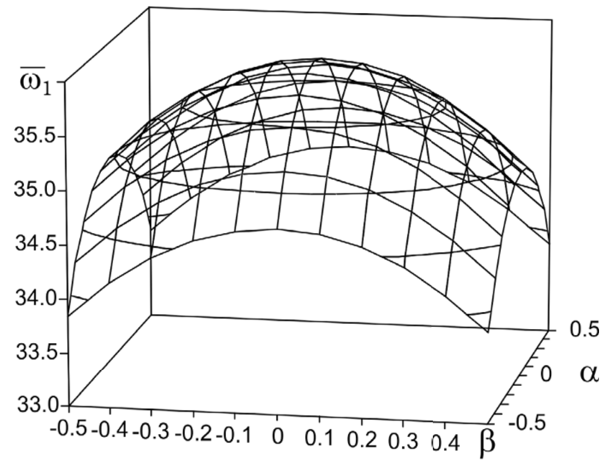

Fig. 7

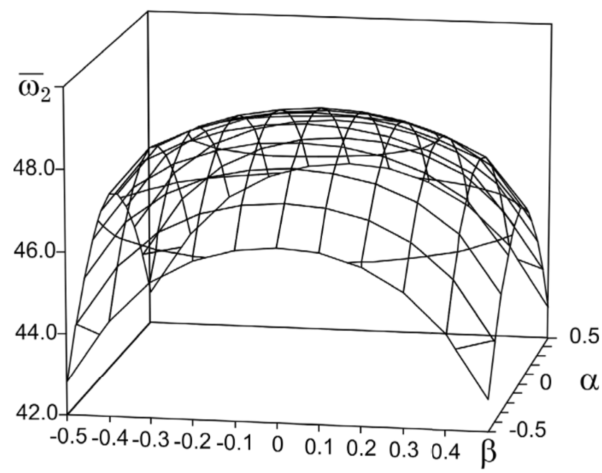

Fig. 8

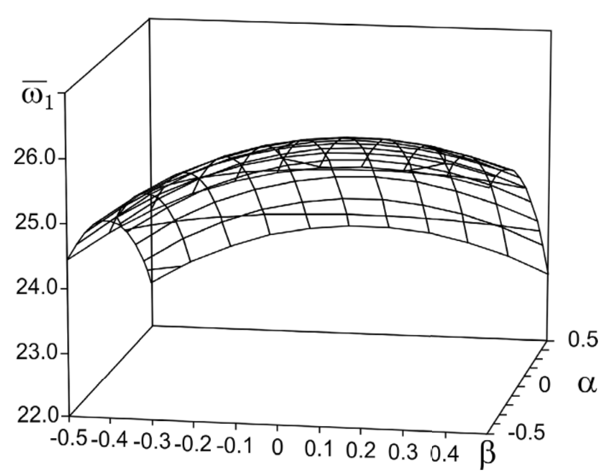

Fig. 9

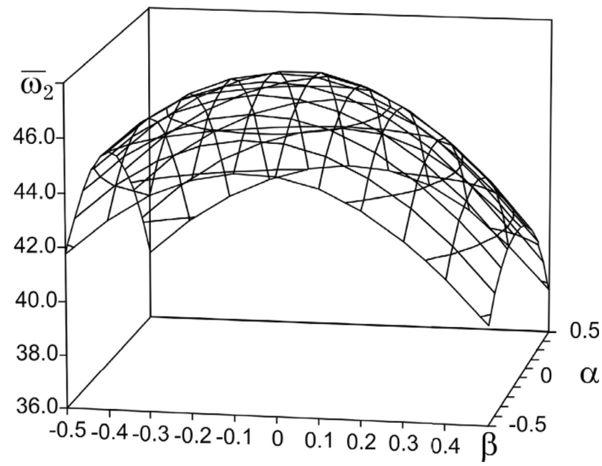

Fig. 10

In Figs. 1–10, we show the plots of dependences of the dimensionless frequencies of free vibrations $\overline{\omega}_i$ of an orthotropic shell whose thickness varies in both coordinate directions on the parameters α and β: the first two frequencies of vibrations on the contours of the shell can be found in Figs. 1 and 2, under the ***BC-1°***-type boundary conditions, in Figs. 3 and 4, under the ***BC-2°***-type boundary conditions, in Figs. 5 and 6, under the ***BC-3°***-type boundary conditions, in Figs. 7 and 8, under the ***BC-4°***-type boundary conditions, and in Figs. 9 and 10, under the ***BC-5°***-type boundary conditions.

The analysis of the plots in Figs. 1–10 enables us to make the following conclusions:

— for all types of boundary conditions, the dependences of the frequencies of vibrations of shallow orthotropic cylindrical shells on the parameters α and β are similar;

— for a given value of the parameter $\alpha = \text{const}$, the frequencies of vibrations of the shells with any value of the parameter β equal in the absolute value and opposite in the sign are almost equal for all considered types of boundary conditions;

— similarly, under the same boundary conditions, in the cases where the parameter $\beta = \text{const}$, the frequencies of free vibrations are almost equal for the values of the parameter α equal in the absolute value but with opposite signs;

— for all analyzed types of boundary conditions, the frequencies of free vibrations of the cylindrical shells form three-dimensional surfaces resembling a paraboloid symmetric about the coordinate planes $\alpha = 0$ and $\beta = 0$;

— the values of the frequencies of free vibrations of cylindrical orthotropic shells with constant thickness ($\alpha = 0$ and $\beta = 0$) are always higher than the corresponding frequencies for shells whose thickness varies according to law (10) for any type of boundary conditions;

— the resonance frequencies of vibrations of the orthotropic shallow shells whose thickness varies in both coordinate directions according to law (10) increase as the absolute value of the parameter α decreases (for constant β) or as the absolute value of the parameter β decreases (for constant α).

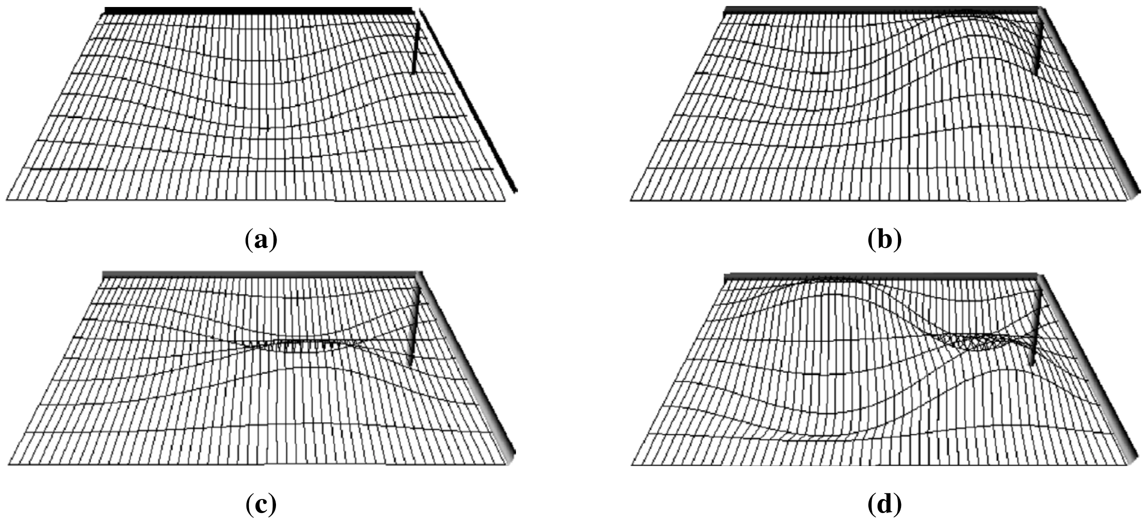

Fig. 11

This means that, for any type of boundary conditions imposed on the contours, the frequencies and modes of free vibrations of orthotropic cylindrical shallow shells can be varied within fairly broad ranges by the proper choice of the values of the parameters of thickness α and β.

In Fig. 11, we show the modes of free vibrations of shallow orthotropic cylindrical shells whose thickness varies in two coordinate directions in the case of rigid fixing of the shells along the contour (***BC-1°***-type boundary conditions) for $\alpha = 0.3$ and $\beta = -0.2$.

The effect of symmetry of the modes of free vibrations of the investigated shells about the planes $x = a/2$, $y = b/2$, and $z = 0$ can be observed. However, it is noticeable for the values of α with large absolute values. The symmetry of the modes of free vibrations is also observed for given $\alpha = \text{const}$ and different β. The centers of the amplitudes of vibrations are shifted in the direction of lower stiffness.

For α and β with equal absolute values but opposite signs, the first modes are symmetric about the plane of the planform, while the higher modes are symmetric about planes passing through the diagonals of the planform of the shells.

In the present work, we develop an efficient numerical method based on the reduction of the system of partial differential equation with variable coefficients corresponding to the mathematical model of the mechanics of free vibrations of anisotropic shallow thick-walled shells whose thickness varies in two coordinate directions within the framework of the refined Timoshenko–Mindlin theory to a system of ordinary differential equations of high order. To solve the obtained system, we use the method of discrete orthogonalization together with the method of step-by-step search. On the basis of the developed approach, we investigate free vibrations of a shallow orthotropic shell whose thickness varies in two coordinate directions. The analysis of the influence of the character of changes in the thickness of the shell on the distribution of its dynamic characteristics is also performed.

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