Solutions for contact problems applied to bevel gears

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Abstract. The article presents solutions to flat contact problems applied to straight-tooth bevel gears, considering both linear and nonlinear dependencies between elastic displacements of the teeth and the resulting stresses. Maximum contact stresses and contact patch dimensions are determined, accounting for both linear and nonlinear relationships between elastic deformations of the teeth and stresses. These solutions are based on a novel contact strength theory. It is demonstrated that the load capacity of bevel gears, considering contact stresses and nonlinear dependencies between elastic deformations and stresses, increases by 1.113 times.

1 Introduction

The role of gear transmissions is exceptionally significant in modern gearbox manufacturing. Currently, the calculation of straight-tooth bevel gears with a flat linear contact system of involute teeth is carried out using the well-known Hertz formula. In this formula, angles δ_1 and δ_2 of the generating (initial) cones of the pinion and gear are introduced, with $\delta_1 + \delta_2 = \delta = 90$ degrees for orthogonal transmissions. Attempts to solve contact problems related to such straight-tooth bevel gears have been made by various researchers at different times. However, these attempts have not yielded the desired results.

Analysis of recent research on this issue is most comprehensively presented in [1, 2]. Solutions to contact problems are based, as mentioned earlier, on the new theory of contact strength of elastically compressed bodies. As for bevel gears, recent developments in this field are also described in patents [3-7].

The aim of this study is to develop practical methods for calculating the contact strength of bevel gears, considering linear and point contact between tooth deformations and the resulting stresses.

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2 Materials and methods

In writing this paper, the authors used an analytical method, with the help of which the studied problems were considered in their unity and development. Taking into account the goals and objectives of the study, the structural and functional method of scientific research was used. The theoretical research was based on the fundamental principles of tribology in general and shipbuilding, utilizing theoretical mechanics approaches, machine components, and principles of design.

3 Results

In books [1, 2], it is demonstrated that when calculating the contact strength of bevel gears, it is advisable to consider transforming the tooth profiles into equivalent straight teeth. As a result, the calculation of the bevel gear engagement can be simplified to the calculation of the equivalent spur gear engagement between the wheels.
In this case, the parameters of the equivalent bevel gears for the pinion d_{v1} and the wheel d_{v2} (Figure 1), as well as the numbers of teeth z_{v1} and z_{v2} are determined based on well-known relationships [1, 2]:

$$d_{v1} = d_1/\cos\delta_1; \qquad d_{v2} = d_2/\cos\delta_2;$$
$$z_{v1} = z_1/\cos\delta_1; \qquad z_{v2} = z_2/\cos\delta_2.$$

where d_1, d_2 are the mean pitch diameters of the pinion and the wheel, z_1, z_2 are the numbers of teeth of the pinion and the wheel; δ_1, δ_2 are the angles of the generating (initial) cones of the pinion and the wheel, and $\delta_1 + \delta_2 = \delta = 90$ degrees.

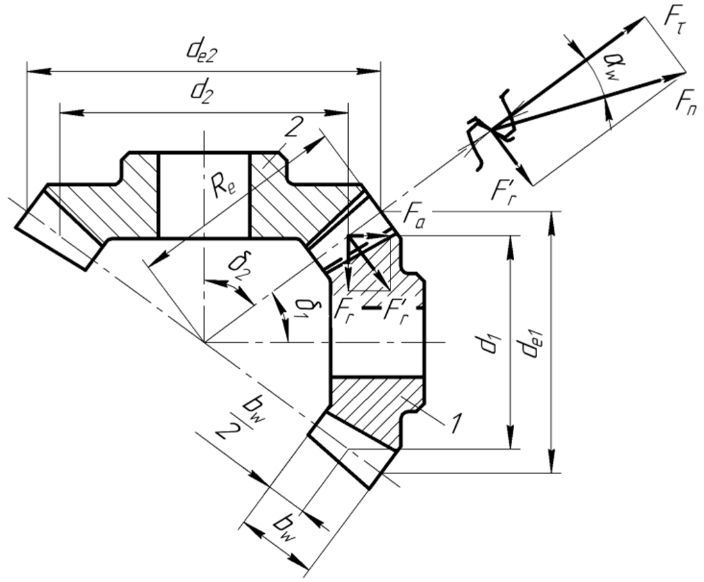

Fig.1. Parameters of bevel gears and forces in engagement: 1 - bevel pinion; 2 - bevel wheel

The calculation model for contact in the considered case is represented by the contact model of two elastically compressed cylinders characterized by radii ρ_1 and ρ_2, which have the following form:

$$\rho_1 = d_1 \frac{\sin\alpha_w}{2\cos\delta_1} = mz_1 \frac{\sin\alpha_w}{2\cos\delta_1};$$

$$\rho_2 = d_2 \frac{\sin\alpha_w}{2\cos\delta_2} = mz_2 \frac{\sin\alpha_w}{2\cos\delta_2}.$$

In accordance with the expressions for ρ_1 and ρ_2, the reduced radius of curvature ρ_w during external linear contact of the cylinders without considering the applied load will have the following form:

$$\frac{1}{\rho_w} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{2\cos\delta_1}{d_1 \sin\alpha_w} + \frac{2\cos\delta_2}{d_2 \sin\alpha_w} = \frac{2}{d_1 \sin\alpha_w}\left(\cos\delta_1 + \frac{\cos\delta_2}{u}\right),$$

where $u = d_2/d_1 = z_2/z_1$ – is the gear ratio; $\cos\delta_1 = \frac{u}{\sqrt{u^2+1}}$; $\cos\delta_2 = \frac{u}{\sqrt{u^2+1}}$.

Taking into account the expressions for $\cos\delta_1$ and $\cos\delta_2$ mentioned above, the aforementioned relationship for determining the reduced radius of curvature ρ_w can be presented in its final form as follows:

$$\frac{1}{\rho_w} = \frac{2}{d_1 \sin\alpha_w}\left(\frac{\sqrt{u^2+1}}{u}\right). \tag{1}$$

First, let's solve the considered problem without taking into account the nonlinearity between elastic deformations and stresses. In this case, as the fundamental relationship establishing the correlation between elastic displacements (deformations) W of the bodies and the stresses σ arising in these bodies, we will adopt the expression [1-3]:

$$W = C_m\sigma^n, \tag{2}$$

where C_m is the dimensional parameter in mm/MPa, $n = 0.7…0.8$ is the power-law exponent characterizing the nonlinearity between elastic displacements of the bodies and the stresses arising in these bodies. In this specific problem, it is assumed to be equal to one, i.e., $n = 1$.

Let's express the average contact stresses σ_m, arising during the interaction of two elastically compressed cylinders as:

$$\sigma_m = \frac{F_n}{2b_0 b_w}, \tag{3}$$

where b_0 is the half-width of the contact patch; b_w is the length of the cylinder or the width of the gear face; F_n is the normal force acting on the elastically compressed bodies.

Let's represent the expression for the total amount of contact deformations (2) of the elastically compressed cylinders 1 and 2 in the following form:

$$W = W_1 + W_2 = C_{m1}\sigma_1 + C_{m2}\sigma_2, \tag{4}$$

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where W_1, W_2 are the elastic deformations of the first and second cylinders; C_{m1}, C_{m2} are the dimensional parameters of the respective cylinders; σ1, σ2 are the contact stresses of the cylinders.

Expressions for contact stresses on each of the contacting surfaces of the bodies, in accordance with Hooke's law, will take the form:

$$\left.\begin{aligned} \sigma_1 &= \frac{\varepsilon E_1}{1-v_1^2} = \frac{E_1}{1-v_1^2}\cdot\frac{\Delta L}{L}; \\ \sigma_2 &= \frac{\varepsilon E_2}{1-v_2^2} = \frac{E_2}{1-v_2^2}\cdot\frac{\Delta L}{L}, \end{aligned}\right\} \tag{5}$$

where v_1, $v2$ are the Poisson's ratios; E_1, E_2 are the young's moduli of the materials of the cylinders; $\varepsilon = \Delta L/L$ is the relative deformation; ΔL is the absolute value of the elastic deformation of the cylinders; ΔL_1, ΔL_2 are the deformations of the first and second cylinders, respectively.

The expression (4), based on equations (5), can be written as two equations, each applying to one of the cylinders:

$$\left.\begin{aligned} W &= 2C_{m1}\sigma_1 = \frac{2C_{m1}E_1}{1-v_1^2}\cdot\frac{\Delta L}{L}; \\ W &= 2C_{m2}\sigma_2 = \frac{2C_{m2}E_2}{1-v_2^2}\cdot\frac{\Delta L}{L}. \end{aligned}\right\} \tag{6}$$

Based on the equality between elastic displacements W and the absolute value of deformation ΔL, i.e., $W = \Delta L$, let's find expressions for the dimensional parameters from equations (6), namely:

$$C_{m1} = \frac{(1-v_1^2)L}{2E_1}; \qquad C_{m2} = \frac{(1-v_2^2)L}{2E_2}.$$

Based on the last two expressions and formula (3), let's determine the relationships for the contact compliance of each of the mating cylinders (teeth):

$$\left.\begin{aligned} \delta_{\text{к}1} &= \frac{C_{m1}\sigma_1}{F_n} = \frac{(1-v_1^2)L}{4b_0 b_w E_1}; \\ \delta_{\text{к}2} &= \frac{C_{m2}\sigma_2}{F_n} = \frac{(1-v_2^2)L}{4b_0 b_w E_2}. \end{aligned}\right\} \tag{7}$$

By multiplying the right sides of expressions (7) by the width of the contact patch $2b_0$, let's determine the coefficients of the beds of the mating bodies:

$$A_1 = \frac{(1-v_1^2)L}{2b_w E_1}; \qquad A_2 = \frac{(1-v_2^2)L}{2b_w E_2}.$$

Taking into account the last two equations, the total coefficient of the bed of elastically compressed cylinders will take the form:

$$A = A_1 + A_2 = \frac{L}{2b_w}\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right). \tag{8}$$

Based on equation (8) for $L = 2b_0$, in which the bed coefficient does not depend on the variable x, the second function of contact deformations $W(x)$ will take the form:

$$W(x) = A\omega(x) = \frac{b_0}{b_w}\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right)\omega(x), \tag{9}$$

where $\omega(x)$ is the load function distributed along the x-axis within the width of the contact patch, N/mm.

Multiplying the right sides of equation (9) and the dependency [2, 3],

$$A\omega(x) = \frac{b_0^2}{2\rho_w}\sqrt{1-\frac{x^2}{b_0^2}}$$

мultiplying by dx, we obtain the expression:

$$\frac{b_0}{b_w}\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right)\omega(x)\,dx = \frac{b_0^2}{2\rho_w}\sqrt{1-\frac{x^2}{b_0^2}}dx,$$

which can be represented as:

$$\frac{b_0}{b_w}\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right)\int_{-b_0}^{b_0}\omega(x)\,dx = \frac{b_0^2}{2\rho_w}\int_{-b_0}^{b_0}\sqrt{1-\frac{x^2}{b_0^2}}dx. \tag{10}$$

The integral on the left side of equation (10) is equal to the compressive force F_n, i.e.,

$$\int_{-b_0}^{b_0}\omega(x)\,dx = F_n.$$

As a result of integrating the right side of equation (10), we can write the equality:

$$\frac{b_0}{b_w}\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right)F_n = \frac{\pi b_0^3}{4\rho_w},$$

from which, we obtain:

$$b_0 = 1{,}129\sqrt{\frac{\rho_w F_n}{b_w}\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right)}. \tag{11}$$

The expression for maximum contact stresses found from the equation

$$\sigma(x) = \frac{b_0\sqrt{b_0^2 - x^2}}{2\rho_w b_w A},$$

at $x = 0$, taking into account dependencies (8) and (11), will take the form:

$$\sigma_H = 0{,}564\sqrt{\frac{F_n}{b_w \rho_w \left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right)}}. \quad (12)$$

If we assume in equations (11) and (12) that v1 = v2 = v and E1 = E2 = E, based on

$$\rho_w = \frac{ud_1 \sin\alpha_w}{2\sqrt{u^2+1}} = \frac{umz_1 \sin\alpha_w}{2\sqrt{u^2+1}}.$$

Then this equation transforms into the form:

$$b_0 = 1{,}076\sqrt{\frac{F_n umz_1 \sin\alpha_w}{b_w E\sqrt{u^2+1}}}; \quad (13)$$

$$\sigma_H = 0{,}591\sqrt{\frac{EF_n\sqrt{u^2+1}}{ub_w mz_1 \sin\alpha_w}}. \quad (14)$$

If the angles of the generating cones δ_1 and δ_2 of the bevel gears are introduced into equations (13) and (14), then these equations will take the form:

$$b_0 = 1{,}076\sqrt{\frac{F_n umz_1 \sin\alpha_w}{b_w E(u\cos\delta_1 + \cos\delta_2)}}; \quad (15)$$

$$\sigma_H = 0{,}591\sqrt{\frac{EF_n(u\cos\delta_1 + \cos\delta_2)}{b_w mz_1 u \sin\alpha_w}}. \quad (16)$$

The derived equations (13) – (16) for finding the half-width of the contact patch b_0 and maximum contact stresses σ_H in the context of straight-tooth traditional bevel gears are presented for the first time, and they have no analogs in modern practices of calculating such transmissions.

Now, let's move on to solving this problem, taking into account the nonlinear relationship between elastic and contact stresses, based on the dependency (2). In this case, the exponent of nonlinearity, n, according to experimental data [1, 2], will be considered as 0,7, i.e., $n = 0{,}7$. Additionally, let's assume

$$\rho_1 = \frac{d_1 \sin\alpha_w}{2\cos\delta_1} \quad \text{and} \quad \rho_2 = \frac{d_2 \sin\alpha_w}{2\cos\delta_2},$$

supposing $m = m_s$.

In order for the deformation W in formula (2) to have the form $W = C_m\sigma^n$ and be expressed in mm, it is necessary for C_m to have the dimensionality of mm/MPa^{0,7}.

Based on the dependency

$$W = \frac{C_m F_n}{2b_0 b_w}$$

Applying it to each of the two elastically compressed bodies, let's find the expressions for contact stresses:

$$\sigma_1 = \frac{E_1}{1-v_1^2}\cdot\frac{\Delta L}{L}; \qquad \sigma_2 = \frac{E_2}{1-v_2^2}\cdot\frac{\Delta L}{L}.$$

Combining the last two expressions with equation (2), let's represent the previously mentioned deformation equations W in the form of two equations, namely:

$$\left.\begin{aligned} W &= 2C_{m1}\left[\frac{E_1}{(1-v_1^2)L}\right]^{0,7}\cdot\Delta L^{0,7};\\ W &= 2C_{m2}\left[\frac{E_2}{(1-v_2^2)L}\right]^{0,7}\cdot\Delta L^{0,7}.\end{aligned}\right\} \tag{17}$$

Then, by multiplying and dividing the right sides of expressions (17) by $\Delta L^{0,3}$, assuming $L = 2b_0$, $\Delta L^{0,3} = (b_0^2/2\rho_w)^{0,3}$ and $W = \Delta L$, let's find expressions for the dimensional parameters:

$$\left.\begin{aligned} C_{m1} &= \frac{1}{2}\left[\frac{2(1-v_1^2)b_0}{E_1}\right]^{0,7}\cdot\frac{b_0^{0,6}}{(2\rho_w)^{0,3}};\\ C_{m2} &= \frac{1}{2}\left[\frac{2(1-v_2^2)b_0}{E_2}\right]^{0,7}\cdot\frac{b_0^{0,6}}{(2\rho_w)^{0,3}}.\end{aligned}\right\} \tag{18}$$

Based on equation (18) and the formula for determining the average contact stresses

$$\sigma_m = \frac{F_n}{2b_0 b_w},$$

let's find the dependencies of the contact compliance of the elastically compressed bodies based on equation (18), considering that the dimension of $\delta_{к1}$ and $\delta_{к2}$ is expressed in mm/N. In this regard, we can write:

$$\left.\begin{aligned} \delta_{к1} &= \frac{W_1}{F_n} = \frac{C_{m1}\sigma_m^{0,7}}{F_n} = 0,406\left(\frac{1-v_1^2}{E_1 b_w}\right)^{0,7}\cdot\frac{b_0^{0,6}}{(\rho_w F_n)^{0,3}};\\ \delta_{к2} &= \frac{W_2}{F_n} = \frac{C_{m2}\sigma_m^{0,7}}{F_n} = 0,406\left(\frac{1-v_2^2}{E_2 b_w}\right)^{0,7}\cdot\frac{b_0^{0,6}}{(\rho_w F_n)^{0,3}}.\end{aligned}\right\} \tag{19}$$

Based on the derived expressions (19), let's write the equation for the bed coefficient A in the form:

$$A = 2b_0(\delta_{к1}+\delta_{к2}) = 0,812\left[\left(\frac{1-v_1^2}{E_1 b_w}\right)^{0,7} + \left(\frac{1-v_2^2}{E_2 b_w}\right)^{0,7}\right]\cdot\frac{b_0^{1,6}}{(\rho_w F_n)^{0,3}}. \tag{20}$$

After substituting the equation (20) into the expression $W(x) = A(x)\omega(x)$, assuming $A(x) = A$ and using the function

$$W(x) = \frac{b_0^2}{2\rho_w}\sqrt{1-\frac{x^2}{b_0^2}},$$

let's represent the dependency

$$A\int_{-b_0}^{b_0}\omega(x)\,dx = AF_n = \frac{0{,}812 b_0^{1,6} F_n}{(\rho_w F_n)^{0,3}}\left\{\left[\frac{(1-\nu_1^2)}{E_1 b_w}\right]^{0,7} + \left[\frac{(1-\nu_2^2)}{E_2 b_w}\right]^{0,7}\right\} = \frac{\pi b_0^3}{4\rho_w},$$

from which we obtain:

$$b_0^{7/5} = 1{,}035\left\{\left[\frac{(1-\nu_1^2)\rho_w F_n}{E_1 b_w}\right]^{0,7} + \left[\frac{(1-\nu_2^2)\rho_w F_n}{E_2 b_w}\right]^{0,7}\right\}.$$

By raising both sides of the last equation to the power of 5/7, let's find the dependency of the half-width of the contact patch, namely:

$$b_0 = 1{,}025\sqrt{\left\{\left[\frac{(1-\nu_1^2)\rho_w F_n}{E_1 b_w}\right]^{0,7} + \left[\frac{(1-\nu_2^2)\rho_w F_n}{E_2 b_w}\right]^{0,7}\right\}^{10/7}}. \qquad (21)$$

Based on dependencies (20) and (21) and the equation [2] $\sigma_H = b_0^2/(2\rho_w \cdot b_w \cdot A)$, let's express the maximum contact stresses in the following form:

$$\sigma_H = 0{,}62\sqrt{\frac{F_n^2}{b_w^2\left\{\left[\frac{(1-\nu_1^2)\rho_w F_n}{E_1 b_w}\right]^{0,7} + \left[\frac{(1-\nu_2^2)\rho_w F_n}{E_2 b_w}\right]^{0,7}\right\}^{10/7}}}. \qquad (22)$$

4 Discussion

In the obtained equations (21) and (22), it is not possible to assess the influence of Poisson's ratios and the moduli of elasticity of materials on the half-width of the contact patch b_0 and the maximum contact stresses σ_H explicitly. However, from equations (21) and (22), it follows that for $n = 0{,}7$, the exponents of the Poisson's ratios and the moduli of elasticity are different from those when $n = 1$ и $\nu_1 \neq \nu_2$, $E_1 \neq E_2$, except for the parameters ρ_w, b_w and F_n, which in all cases are characterized by exponents equal to 1/2.

In transmissions, as is known, steels are used as materials for manufacturing gears with $\nu_1 = \nu_2$ and $E_1 = E_2$. In connection with this, assuming $\nu_1 = \nu_2 = \nu = 0{,}3$ and $E_1 = E_2$, let's transform the dependencies (21) and (22) into a more simplified form:

$$b_0 = 1{,}135\sqrt{\frac{F_n u m z_1 \sin\alpha_w}{b_w E\sqrt{u^2+1}}}; \qquad (23)$$

$$\sigma_H = 0{,}56\sqrt{\frac{EF_n\sqrt{u^2+1}}{b_w umz_1 \sin\alpha_w}}. \tag{24}$$

Equations (23) and (24), like the previously derived equations (15) and (16), are found taking into account that the reduced radius of curvature has the form:

$$\rho_w = \frac{umz_1 \sin\alpha_w}{2\sqrt{u^2+1}}.$$

On the other hand, as can be inferred from the previously provided data, the mentioned radius of curvature can be expressed through the angles of the generating (initial) cones δ_1 and δ_2 as:

$$\rho_w = \frac{umz_1 \sin\alpha_w}{2(u\cos\delta_1 + \cos\delta_2)}.$$

In accordance with the given expression for the radius ρ_w, the expressions (21) and (22) for $v_1 = v_2 = v = 0{,}3$ and $E_1 = E_2 = E$ take the form:

$$b_0 = 1{,}135\sqrt{\frac{F_n umz_1 \sin\alpha_w}{b_w E(u\cos\delta_1 + \cos\delta_2)}}; \tag{25}$$

$$\sigma_H = 0{,}56\sqrt{\frac{EF_n(u\cos\delta_1 + \cos\delta_2)}{b_w mz_1 u \sin\alpha_w}}. \tag{26}$$

Comparing the numerical coefficients for b_0 and σ_H in formulas (15), (16) and (25), (26), considering the nonlinearity between elastic deformations and stresses, the half-width of the contact patch b_0 increases by a factor of 1,135/1,076 = 1,055, and the maximum contact stresses σ_H consequently decrease by a factor of 0,591/0,56 = 1,055.

The reduction in contact stresses by a factor of 1,055 is equivalent, as known, to an increase in load-carrying capacity by this stress factor squared, i.e., approximately by 11.3%.

5 Conclusions

1. For the first time, solutions to flat contact problems have been developed for straight-toothed bevel gears, determining maximum contact stresses and contact patch sizes, considering linear and nonlinear dependencies between elastic deformations and stresses.

2. It is shown that in the case of nonlinear dependence between elastic deformations and stresses, maximum contact stresses decrease by a factor of 1,055, which corresponds to an increase in load-carrying capacity by a factor of 1,113 compared to linear dependence between these deformations and stresses.

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