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INVESTIGATING THE STABILITY OF OSCILLATIONS OF RECTANGULAR PLATES IN AN INFINITELY LONG RECTANGULAR PARALLELEPIPED WITH AN IDEAL FLUID

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The object of this study is two thin elastic isotropic rectangular plates in an infinitely long rectangular parallelepiped with an ideal fluid. The first plate is the upper base of the rectangular parallelepiped, and the second one horizontally separates ideal fluids that have different densities. The subject of the study is the normal joint plane vibrations of elastic rectangular plates and an incompressible fluid and the conditions that enable the stability of these vibrations.

In the linear statement, the frequency spectrum of normal plane vibrations of two elastic isotropic plates in an infinitely long rectangular parallelepiped with an ideal incompressible fluid has been investigated. The frequency equation of joint vibrations of the plates and the ideal fluid was reduced to the form of an eighth-order determinant for arbitrary cases of fixing the contours of the plates. The case of clamped contours of the plates and the case of rebirth of the plates into membranes is analyzed. Based on analytical studies of infinite series in the transcendental frequency equation, exact stability conditions for the combined oscillations of plates and liquid were established. It has been shown that instability of oscillations of plates and liquid occurs when a heavier liquid is above a less heavy liquid. The derived stability conditions for symmetric and asymmetric oscillations of plates and liquid do not depend on the elastic parameters of the upper plate, the mass characteristics of the plates and the depths of filling liquids. The analytically obtained exact stability conditions for the combined oscillations of the plate and liquid generalize the previously obtained approximate stability conditions for this problem. The numerical calculations of the frequency equation confirmed the analytical studies of the stability conditions. The results could be used in the calculation and design of mechanical objects related to the storage and transportation of liquid cargo

Keywords: rectangular plates, ideal fluid, infinitely long rectangular parallelepiped, plane vibrations, stability

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1. Introduction

In modern tanker, railroad, aviation, as well as rocket and space technology, and in other branches of the national economy, structures are actively used that have the form of elastic compartments with a liquid or liquids with different densities. The instability of oscillations of elastic compartments with a liquid leads to the destruction of structures. To establish stability conditions, it is necessary to solve a very complex hydroelasticity problem and currently there are only individual solutions to this problem. One of the most effective approaches to simplifying this complex hydroelastic problem and deriving analytical solutions is an approach based on considering plane oscillations of elastic rectangular plates and a liquid. This makes it possible to estimate with accuracy reasonable for practice the critical values of mechanical parameters at which instability of joint oscillations of plates and liquid occurs. Various types of oscillations of liquids and plates during the transportation of substances of different densities in tanks affect the stability of the system, which can lead to a violation of the stability of the system, and subsequently to destruction.

Therefore, it is a relevant task to study the influence of combined oscillations of elastic plates and a liquid on the frequency spectrum and on the stability of oscillations of a mechanical system. At the same time, one of the most effective approaches to simplifying this complex hydroelastic problem and obtaining analytical solutions is the approach based on considering plane oscillations of elastic rectangular plates and a liquid. This makes it possible to investigate mechanical regularities with accuracy reasonable for practice and to establish stability conditions convenient for calculations.

2. Literature review and problem statement

The problem of normal oscillations of a membrane on the free surface of a liquid in a long rectangular parallelepiped was considered in [1, 2]. The studies were carried out on the basis of the Lagrange–Euler approach, but the stability issues were not considered. In papers [3, 4], exact stability conditions for joint oscillations of a plate and a liquid in an infinitely long rectangular parallelepiped with a rigid bottom [3] and an elastic bottom [4] were established. These stability conditions were obtained on the basis of analytical solutions of the roots of an infinite series. For limited sizes of elastic containers with a liquid, only numerical approaches are available in the literature [5, 6]. A fairly large number of works have tackled problems describing the natural oscillations of an ideal liquid in a straight circular cylinder with elastic bases, in particular [7, 8]. In paper [7], oscillations of an ideal liquid in a circular cylindrical tank with elastic bases in the form of circular plates are considered. An analytical method based on the Fourier-Bessel series expansion and the Rayleigh-Ritz method is proposed. Work [8] reports a study on the frequency equations of asymmetric and symmetric natural oscillations of an ideal two-layer fluid in a rigid circular cylindrical tank with an elastic top and bottom in the form of clamped circular plates. Using the example of a homogeneous fluid with a free surface and an elastic bottom in the form of a membrane, the frequency spectrum of joint oscillations was analyzed analytically and numerically, but the issues of stability of the joint oscillations of the plate and fluid were not considered.

In paper [9], a solution to the hydroelastic problem of free oscillations of a thin isotropic plate horizontally separating ideal incompressible fluids of different densities in a rigid cylindrical tank of arbitrary cross-section was constructed. To solve the complicated inhomogeneous biharmonic equation, the fundamental system of solutions of the biharmonic equation (FSR) and the eigenforms of oscillations of an ideal fluid in a cylindrical cavity were applied. Using the example of a clamped plate, the frequency equation was simplified by decomposing the homogeneous biharmonic equation into two harmonic equations and using Green's formula for the Laplace operator. It was shown that in this case the frequency equation does not depend on FSR, which significantly simplified this equation, since the FSR depends on the unknown frequency. It should also be noted that the derived equation has a single form for the cases of a straight circular cylinder and a rectangular channel and for some cases coincides with the previously built equations; however, the issues of stability of the combined oscillations of the plate and the liquid were not considered in [7–9]. In [10], the problem of oscillations of a solid body with a liquid under the action of a spring force (Sretensky problem) and the problem of oscillations of a physical pendulum are generalized to the case of a multilayer ideal liquid separated by elastic plates. From the positive definiteness of the potential energy (Sretensky problem) and the changed potential energy (physical pendulum), the conditions for the stability of the equilibrium position are established. More detailed studies were carried out for a cylindrical cavity of arbitrary cross section. It is shown that in the Sretensky problem, for the stability of the equilibrium position, it is necessary and sufficient that the equilibrium position of elastic plates and liquid in a stationary solid body be stable, and it is sufficient that the heavier liquid be below the less heavy one. In the problem of oscillations of a physical pendulum, for the stability of the equilibrium position, it is also necessary that the equilibrium position of elastic rectangular plates and incompressible liquid in a stationary solid body be stable. It was proved that the preliminary tension of the plates makes it possible to stabilize the unstable equilibrium position of the physical pendulum. It should be noted that in that work, the conditions for the stability of oscillations of a solid body, plates, and liquid were established on the condition of positive definiteness of potential energy.

Of the latest works on this topic, paper [11] should be noted, which considers a more complex hydroelastic problem of spatial oscillations of a liquid in an elastic rectangular parallelepiped.

From our review of the literature it follows that the issues of stability of vibrations of rectangular plates in an infinitely long rectangular parallelepiped with an ideal fluid have not been resolved since to obtain exact stability conditions it was necessary to involve new analytical methods for studying stability.

3. The aim and objectives of the study

The aim of our work is to analytically solve the plane hydroponic problem of oscillations and stability of a thin isotropic rectangular plate that separates ideal incompressible fluids of different densities in a rigid rectangular channel with an elastic upper plate. For this purpose, it is necessary to derive the frequency equation, simplify it, and analytically establish the stability conditions. This will make it possible to take into account the influence of the elastic and mass characteristics of the plates and the fluid on the frequency spectrum and stability of joint oscillations.

To achieve the goal, the following tasks were set:

– to derive the integrated-differential equations of joint oscillations of rectangular plates and an ideal fluid;

– to construct the frequency equation of normal joint oscillations of rectangular;

– plates and an ideal fluid for general cases of fixing the contours of the plates and simplify this frequency equation for the case of clamped contours of the plates and membranes;

– based on analytical studies of infinite series in the transcendental frequency equation, establish exact stability conditions for coupled oscillations of rectangular plates or membranes and an ideal fluid, and to verify the resulting stability conditions, perform numerical calculations of the frequency equation for the case of a membrane.

4. The study materials and methods

The object of our study is two thin elastic isotropic rectangular plates in an infinitely long rectangular parallelepiped with an ideal fluid. The mechanical system consists of two thin elastic isotropic rectangular plates in an infinite rectangular parallelepiped. The first plate is the upper base of the rectangular parallelepiped, and the second one horizontally separates ideal fluids of different densities ρ_i (i=1, 2). The long rectangular parallelepiped has width b, where (b=2a). The plates are considered isotropic, with constant bending stiffness D_i and with forces T_i in the middle surface (i=1, 2). The index i=1 will correspond to the upper plate, and i=2 to the inner one (Fig. 1). The contours of the plates are arbitrarily fixed. The liquid located above the density ρ_1 occupies space in the channel to depth h_1, and the ideal liquid located below to depth h_2.

The $Oxyz$ coordinate system is located in such a way that the Oxy plane is on the undisturbed median surface of the inner plate, the Oy axis is directed along an infinitely long rectangular parallelepiped, and the Oz axis is opposite to the gravitational acceleration vector $\vec{g}$ (Fig. 1).

The oscillations of the plates and the liquid are considered in a linear statement, the joint oscillations of the plates and the liquid are considered continuous, and the motion of the liquids is considered potential. The forms of the plate deflection are given as the sum of the fundamental solutions to the homogeneous equation for each plate and a partial solution to the inhomogeneous equation in the form of an expansion in terms of the eigenfunctions of oscillations of an ideal liquid in a rectangular channel. A new method was proposed for the study, which is associated with the analytical determination of the critical values of parameters at which the frequency of joint oscillations approaches zero.

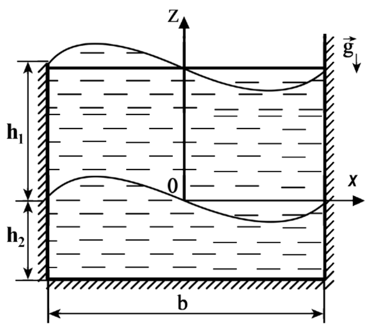

Fig. 1. Cross-section of an infinite rectangular parallelepiped with an elastic upper and rigid lower base, containing fluids of different densities separated by an elastic plate

5. Results of research on the stability of vibrations of rectangular plates in an infinitely long rectangular parallelepiped

5. 1. Basic integrated-differential equations of joint vibrations of rectangular plates and an ideal fluid

The equations of plane vibrations of elastic rectangular plates and an ideal fluid will take the form [3, 4]:

$$k_{0i}\frac{\partial^2 W_i}{\partial t^2}+D_i\frac{\partial^4 W_i}{\partial x^4}-T_i\frac{\partial^2 W_i}{\partial x^2}=P_i-P_{i-1} \text{ at } z=z_i,\ (i=1,2), \quad (1)$$

$$\frac{\partial^2 \Phi_i}{\partial x^2}+\frac{\partial^2 \Phi_i}{\partial z^2}=0,\ (i=1,2), \quad (2)$$

under boundary conditions:

$$\frac{\partial W_1}{\partial t}=\frac{\partial \Phi_1}{\partial z} \text{ at } z=h_1,$$

$$\frac{\partial W_2}{\partial t}=\frac{\partial \Phi_1}{\partial z}=\frac{\partial \Phi_2}{\partial z} \text{ at } z=0, \quad (3)$$

$$\frac{\partial \Phi_2}{\partial z}=0 \text{ at } z=-h_2, \quad (4)$$

$$\left(\mathfrak{L}_{ijp}\left[W_i\right]\right)\Big|_{\gamma_j}=0,\ (i,j,p=1,2), \quad (5)$$

$$\int_{-a}^{a} W_i \mathrm{d}x=0, \quad (6)$$

$$\left.\frac{\partial \Phi_i}{\partial x}\right|_{\gamma_j}=0,\ (j,p=1,2), \quad (7)$$

where $k_{0i}=\rho_{0i}\cdot h_{0i}$; $W_i(x, t)$, ρ_{0i}, h_{0i} – normal deflection, density, and thickness of the i-th plate; $\Phi_i(x,y,t)$ – velocity potential of the i-th fluid (i=1, 2); $P_i(x, z, t)$ – hydrodynamic pressure in the i-th fluid, and $P_0(x, t)$ – pressure above the upper base; $z_i=h_1$ at i=1 and 0 at i=2; $\mathfrak{L}_{ijp}$ – differential operators of the boundary conditions of fixing the plate on the contour γ_j.

Taking into account the Cauchy-Lagrange integral, equation (1) can be written as follows:

$$k_{0i}\frac{\partial^2 W_i}{\partial t^2}+D_i\frac{\partial^4 W_i}{\partial x^4}-T_i\frac{\partial^2 W_i}{\partial x^2}+g\Delta\rho_i W_i=\rho_{i-1}\frac{\partial \Phi_{i-1}}{\partial t}-$$
$$-\rho_i\frac{\partial \Phi_i}{\partial t}+Q_i\rho_i-Q_{i-1}\rho_{i-1}-\left(P_0+gh_1\right)\delta_{i1}, \quad (8)$$

at $z=z_i,\ (i=1,2)$,

where $\Delta\rho_i=\rho_i-\rho_{i-1}$ ($\rho_0=0$); Q_i is an arbitrary function of time; δ_{i1} is the Kronecker symbol.

Let the functions $\Phi_i(x,z,t)$ take the form:

$$\Phi_i=\sum_{n=1}^{\infty}\left[A_{in}(t)\mathrm{e}^{k_n z}+B_{in}(t)\mathrm{e}^{-k_n z}\right]\psi_n(x),\ (i=1,2), \quad (9)$$

where the functions $\psi_n(x)=\cos k_n(x+a)$, and their corresponding eigenvalues $k_n=\pi n/2a$.

Expression (9) satisfies equation (2) and boundary condition (7).

From (9), (3), (4) and the orthogonality of the functions ψ_n, the following linear system with respect to unknowns A_{in}, B_{in} (i=1, 2) follows:

$$A_{1n}e^{\kappa_{1n}}-B_{1n}e^{-\kappa_{1n}}=\frac{1}{k_n}\dot{W}_{1n},\quad A_{1n}-B_{1n}=\frac{1}{k_n}\dot{W}_{2n},$$

$$A_{1n}-B_{1n}=A_{2n}-B_{2n},\quad A_{2n}e^{-\kappa_{2n}}-B_{2n}e^{\kappa_{2n}}=0. \quad (10)$$

System (10) can be solved as follows:

$$A_{1n}=\frac{\dot{W}_{1n}-\dot{W}_{2n}e^{-\kappa_{1n}}}{2k_n\sinh\kappa_{1n}},\quad B_{1n}=\frac{\dot{W}_{1n}-\dot{W}_{2n}e^{\kappa_{1n}}}{2k_n\sinh\kappa_{1n}},$$

$$A_{2n}=\frac{\dot{W}_{2n}e^{\kappa_{2n}}}{2k_n\sinh\kappa_{2n}},\quad B_{2n}=\frac{\dot{W}_{2n}e^{-\kappa_{2n}}}{2k_n\sinh\kappa_{2n}}. \quad (11)$$

Here:

$$W_{in}=\frac{1}{N_n^2}\int_{-a}^{a}W_i\psi_n\mathrm{d}x,\quad N_n^2=\int_{-a}^{a}\psi_n^2\mathrm{d}x=a,\quad \kappa_{in}=h_i k_n. \quad (12)$$

Taking into account ratios (9), (11), (12), equation (8) will take the form:

$$k_{01}\frac{\partial^2 W_1}{\partial t^2}+D_1\frac{\partial^4 W_1}{\partial x^4}-T_1\frac{\partial^2 W_1}{\partial x^2}+g\rho_1 W_1=$$
$$=\sum_{n=1}^{\infty}\frac{-a_{1n}\ddot{W}_{1n}+b_n\ddot{W}_{2n}}{k_n}\psi_n+Q_1\rho_1-P_0-gh_1, \quad (13)$$

$$k_{02}\frac{\partial^2 W_2}{\partial t^2}+D_2\frac{\partial^4 W_2}{\partial x^4}-T_2\frac{\partial^2 W_2}{\partial x^2}+g\Delta\rho W_2=$$
$$=\sum_{n=1}^{\infty}\frac{b_n\ddot{W}_{1n}-a_{2n}\ddot{W}_{2n}}{k_n}\psi_n+Q_2\rho_2-Q_1\rho_1, \quad (14)$$

where:

$$b_n = \rho_1 \big/ \sinh \kappa_{1n},\quad a_{in} = \rho_i \coth \kappa_{in} + \rho_{i-1} \coth \kappa_{i-1,n},$$

$$a_{1n} = \rho_1 \coth \kappa_{1n},\quad a_{2n} = a_n = \rho_1 \coth \kappa_{1n} + \rho_2 \coth \kappa_{2n}.$$

Thus, the joint oscillations of elastic rectangular plates and an ideal fluid are found from the system of integrated-differential equations (12) to (14), boundary conditions (5), conditions for the conservation of the volume of an incompressible fluid (6) and given initial conditions.

5. 2. Normal frequencies of combined vibrations of rectangular plates and an ideal fluid

Assume:

$$W_i(x,t) = w_i(x)e^{i\omega t},\quad Q_i\rho_i = \tilde{C}_i e^{i\omega t},\quad (i=1,2),\ P_0 = 0. \tag{15}$$

From (12) to (15), boundary conditions (5), and condition (6), it follows:

$$\frac{d^4 w_i}{dx^4} - p_i \frac{d^2 w_i}{dx^2} + q_i w_i = \\ = \frac{\omega^2}{D_i}\sum_{n=1}^{\infty} \frac{a_{in} w_{in} - b_n w_{3-i,n}}{k_n}\psi_n + C_i - C_{i-1} + \tilde{h}_1 \delta_{i1}, \tag{16}$$

$$w_{in} = \frac{1}{a}\int_{-a}^{a} w_i \psi_n \mathrm{d}x,\quad \left(i = \overline{1,2}\right), \tag{17}$$

$$\int_{-a}^{a} w_i \mathrm{d}x = 0, \tag{18}$$

$$\left(\mathfrak{L}_{ijp}\left[w_i\right]\right)\Big|_{\gamma_j} = 0,\quad (i,j,p = 1,2), \tag{19}$$

Here $C_i = \tilde{C}_i / D_i$, $\tilde{h}_1 = -gh_1 / D_1$.

The solution to equation (16) is represented as a linear combination of four fundamental solutions w_{ik}^0 $\left(i=1,2,\ k=\overline{1,4}\right)$ to the corresponding homogeneous equation:

$$\frac{d^4 w_{ik}^0}{dx^4} - p_i \frac{d^2 w_{ik}^0}{dx^2} + q_i w_{ik}^0 = 0, \tag{20}$$

and partial solution to inhomogeneous equation (16) [3–5, 11–14]:

$$w_i = \sum_{k=1}^{4} A_{ik}^0 w_{ik}^0 + \sum_{n=1}^{\infty} \tilde{C}_{in}\psi_n + w_{0i} + \tilde{h}_1 \delta_{i1},\quad \left(i=\overline{1,2}\right), \tag{21}$$

where $\rho_i = T_i/D_i$, $q_i = (g\Delta\rho_i - k_{0i}\omega^2)/D_i$, A_{ik}^0, $\tilde{C}_{in}$ and w_{0i} are unknown constants.

Taking into account the representation of (21) and relations $\frac{d^2\psi_n}{dx^2} = -k_n^2\psi_n$, $\frac{d^4\psi_n}{dx^4} = k_n^4\psi_n$, the unknown coefficients $\tilde{C}_{1n}$ and $\tilde{C}_{2n}$ take the form:

$$\tilde{C}_{1n} = \omega^2 \frac{a_{1n}w_{1n} - b_n w_{2n}}{k_n d_{1n}},\ \tilde{C}_{2n} = -\omega^2 \frac{b_n w_{1n} - a_{2n} w_{2n}}{k_n d_{2n}}. \tag{22}$$

Here $d_{in} = \left(D_i k_n^2 + T_i\right)k_n^2 + g\Delta\rho_i - k_{0i}\omega^2$.

Taking into account (21) and (22), the system of linear equations for determining w_{1n} and w_{2n} follows:

$$\begin{cases} w_{1n} = \sum\limits_{k=1}^{4} A_{1k}^0 E_{kn}^0 + \dfrac{\omega^2}{k_n d_{1n}}\left(a_{1n}w_{1n} - b_n w_{2n}\right), \\ w_{2n} = \sum\limits_{k=1}^{4} A_{2k}^0 E_{kn}^0 - \dfrac{\omega^2}{k_n d_{2n}}\left(b_n w_{1n} - a_{2n} w_{2n}\right), \end{cases}$$

The solution to which is as follows:

$$\begin{cases} w_{1n} = \dfrac{1}{\tilde{\Delta}_n}\left[\begin{array}{l}\left(1 - \omega^2 \dfrac{a_{2n}}{k_n d_{2n}}\right)\sum\limits_{k=1}^{4} A_{1k}^0 E_{1kn}^0 - \\ -\omega^2 \dfrac{b_n}{k_n d_{1n}}\sum\limits_{k=1}^{4} A_{2k}^0 E_{2kn}^0\end{array}\right], \\ w_{2n} = \dfrac{1}{\tilde{\Delta}_n}\left[\begin{array}{l} -\omega^2 \dfrac{b_n}{k_n d_{2n}}\sum\limits_{k=1}^{4} A_{1k}^0 E_{1kn}^0 + \\ + \left(1 - \omega^2 \dfrac{a_{1n}}{k_n d_{1n}}\right)\sum\limits_{k=1}^{4} A_{2k}^0 E_{2kn}^0\end{array}\right], \end{cases} \tag{23}$$

where:

$$\tilde{\Delta}_n = \left(1 - \omega^2 \frac{a_{1n}}{k_n d_{1n}}\right)\left(1 - \omega^2 \frac{a_{2n}}{k_n d_{2n}}\right) - \omega^4 \frac{b_n^2}{k_n^2 d_{1n} d_{2n}},$$

$$E_{ikn}^0 = \frac{1}{N_n^2}\int_{-a}^{a} w_{ik}^0 \psi_n \mathrm{d}x,\quad \left(k = \overline{1,4}\right). \tag{24}$$

Expression (21) for the shape of the deflection of plates w_1 and w_2, taking into account (18), (22)m and (23), will take the form:

$$\begin{cases} w_1 = \sum\limits_{k=1}^{4}\left[\begin{array}{l}\left(w_{1k}^0 - \tilde{w}_{1k}^0 + \sum\limits_{n=1}^{\infty} a_{11n} E_{1kn}^0 \psi_n\right) A_{1k}^0 + \\ + \left(\sum\limits_{n=1}^{\infty} a_{12n} E_{2kn}^0 \psi_n\right) A_{2k}^0\end{array}\right], \\ w_2 = \sum\limits_{k=1}^{4}\left[\begin{array}{l}\left(\sum\limits_{n=1}^{\infty} a_{21n} E_{1kn}^0 \psi_n\right) A_{1k}^0 + \\ + \left(w_{2k}^0 - \tilde{w}_{2k}^0 + \sum\limits_{n=1}^{\infty} a_{22n} E_{2kn}^0 \psi_n\right) A_{2k}^0\end{array}\right], \end{cases} \tag{25}$$

Here:

$$\tilde{w}_{ik}^0 = \frac{1}{2a}\int_{-a}^{a} w_{ik}^0 \mathrm{d}x,\quad a_{11n} = \omega^2 \frac{\left(k_n d_{2n} - \omega^2 a_{2n}\right)a_{1n} + \omega^2 b_n^2}{\Delta_n},$$

$$a_{12n} = -\omega^2 \frac{k_n d_{2n} b_n}{\Delta_n},\quad a_{21n} = -\omega^2 \frac{k_n d_{1n} b_n}{\Delta_n},$$

$$a_{22n} = \omega^2 \frac{\left(k_n d_{1n} - \omega^2 a_{1n}\right)a_{2n} + \omega^2 b_n^2}{\Delta_n},$$

$$\Delta_n = \left(k_n d_{1n} - \omega^2 a_{1n}\right)\left(k_n d_{2n} - \omega^2 a_{2n}\right) - b_n^2 \omega^4. \tag{26}$$

Thus, the deflection forms of the plates w_1 and w_2 will be written as follows:

$$w_i = \sum_{l=1}^{2}\sum_{k=1}^{4}\left[\left(w_{ik}^0 - \tilde{w}_{ik}^0\right)\delta_{il} + \sum_{n=1}^{\infty} a_{iln} E_{lkn}^0 \psi_n\right] A_{lk}^0,\quad \left(i = \overline{1,2}\right), \tag{27}$$

where δ_{il} is the Kronecker symbol.

In the case of membranes (D_i=0) in expressions (27) it is necessary to assume k=1,2.

From boundary conditions (19) it follows:

$$\sum_{l=1}^{2}\sum_{k=1}^{4}\left(\mathfrak{L}_{ijpk}^0 \delta_{il} + \sum_{n=1}^{\infty} a_{iln} E_{lkn}^0 \mathfrak{L}_{ijpn}\right) A_{lk}^0 = 0,\quad (i,j,p = 1,2), \tag{28}$$

Here:

$$\mathfrak{L}^0_{ijpk}=\left(\mathfrak{L}_{ijp}\left[w^0_{ik}-\tilde{w}^0_{ik}\right]\right)\Big|_{\gamma_j},\quad \mathfrak{L}_{ijpn}=\left(\mathfrak{L}_{ijp}\left[\psi_n\right]\right)\Big|_{\gamma_j}. \qquad (29)$$

From the system of linear equations (28), the frequency equation of the joint eigenoscillations of the plates and the liquid is:

$$\left|\left\|C_{qr}\right\|^8_{q,r=1}\right|=0, \qquad (30)$$

where the values of coefficients C_{qr} are found from formula (28).

Thus, the problem under consideration has a discrete spectrum of eigenvalues ω_l^2, and the corresponding eigenfunctions w_{il} are found from the linear system (28) and relations (27) and form a complete and orthogonal system of functions on the interval $[-a, a]$.

In the following, the main attention will be paid to the case of pinched contours since it is most often encountered in practice. In this case, the coefficients of the determinant of the frequency equation (30) take the following form:

$$C_{1k}=B_{11k}+\sum_{n=1}^{\infty}a_{11n}E^0_{1kn},\quad C_{1,k+4}=\sum_{n=1}^{\infty}a_{12n}E^0_{2kn},$$

$$C_{2,k+4}=0,\ C_{2k}=C^0_{11k},$$

$$C_{3k}=B_{12k}+\sum_{n=1}^{\infty}a_{11n}E^0_{1kn}\left(-1\right)^n,\ C_{3,k+4}=\sum_{n=1}^{\infty}a_{12n}E^0_{2kn}\left(-1\right)^n,$$

$$C_{4k}=C^0_{12k},\quad C_{4,k+4}=0,$$

$$C_{5k}=\sum_{n=1}^{\infty}a_{21n}E^0_{1kn},C_{5,k+4}=B_{21k}+\sum_{n=1}^{\infty}a_{22n}E^0_{2kn},$$

$$C_{6k}=0,\quad C_{6,k+4}=C^0_{21k}. \qquad (31)$$

$$C_{7k}=\sum_{n=1}^{\infty}a_{21n}E^0_{1kn}\left(-1\right)^n,\quad C_{7,k+4}=B_{22k}+\sum_{n=1}^{\infty}a_{22n}E^0_{2kn}\left(-1\right)^n,$$

$$C_{8k}=0,\quad C_{8,k+4}=C^0_{22k},\quad \left(k=\overline{1,4}\right),$$

Here:

$$B_{ijk}=\left(w^0_{ik}-\tilde{w}^0_{ik}\right)\Big|_{\gamma_j},\quad C^0_{ijk}=\frac{\mathrm{d}w^0_{ik}}{\mathrm{d}x}\Big|_{\gamma_j}.$$

Let the functions w^0_{ik} be represented as a series over a complete and orthogonal system of eigenfunctions ψ_n [3], then the coefficients (31) will be written as follows:

$$C_{1k}=\sum_{n=1}^{\infty}\tilde{a}_{11n}E^0_{1kn},\quad C_{1,k+4}=\sum_{n=1}^{\infty}a_{12n}E^0_{2kn},$$

$$C_{2k}=C^0_{11k},\quad C_{2,k+4}=0,$$

$$C_{3k}=\sum_{n=1}^{\infty}\tilde{a}_{11n}E^0_{1kn}\left(-1\right)^n,\quad C_{3,k+4}=\sum_{n=1}^{\infty}a_{12n}E^0_{2kn}\left(-1\right)^n,$$

$$C_{4k}=C^0_{12k},\quad C_{4,k+4}=0,$$

$$C_{5k}=\sum_{n=1}^{\infty}a_{21n}E^0_{1kn},\quad C_{5,k+4}=\sum_{n=1}^{\infty}\tilde{a}_{22n}E^0_{2kn},$$

$$C_{6k}=0,\quad C_{6,k+4}=C^0_{21k},$$

$$C_{7k}=\sum_{n=1}^{\infty}a_{21n}E^0_{1kn}\left(-1\right)^n,\quad C_{7,k+4}=\sum_{n=1}^{\infty}\tilde{a}_{22n}E^0_{2kn}\left(-1\right)^n,$$

$$C_{8k}=0,\quad C_{8,k+4}=C^0_{22k}. \qquad (32)$$

Here:

$$\tilde{a}_{11n}=1+a_{11n}=\frac{\left(k_nd_{2n}-\omega^2a_{2n}\right)k_nd_{1n}}{\Delta_n},$$

$$\tilde{a}_{22n}=1+a_{22n}=\frac{\left(k_nd_{1n}-\omega^2a_{1n}\right)k_nd_{2n}}{\Delta_n},\quad \left(k=\overline{1,4}\right).$$

For a membrane (D_i=0), expression (30) will take the form:

$$\left|\left\|C_{qr}\right\|^4_{q,r=1}\right|=0, \qquad (33)$$

and the coefficients (31) will be written as follows:

$$C_{1k}=\sum_{n=1}^{\infty}\tilde{a}_{11n}E^0_{1kn},\quad C_{1,k+2}=\sum_{n=1}^{\infty}a_{12n}E^0_{2kn},$$

$$C_{2k}=\sum_{n=1}^{\infty}\tilde{a}_{11n}E^0_{1kn}\left(-1\right)^n,\quad C_{2,k+2}=\sum_{n=1}^{\infty}a_{12n}E^0_{2kn}\left(-1\right)^n,$$

$$C_{3k}=\sum_{n=1}^{\infty}\tilde{a}_{21n}E^0_{2kn},\quad C_{3,k+2}=\sum_{n=1}^{\infty}a_{22n}E^0_{2kn},$$

$$C_{4k}=\sum_{n=1}^{\infty}\tilde{a}_{21n}E^0_{1kn}\left(-1\right)^n,$$

$$C_{4,k+2}=\sum_{n=1}^{\infty}\tilde{a}_{22n}E^0_{2kn}\left(-1\right)^n,\quad \left(k=\overline{1,2}\right). \qquad (34)$$

By performing transformations with the rows and columns of the determinant of equation (30), as was done in [3, 4], this equation can be written in a single form for symmetric and asymmetric frequencies:

$$\left(\sum_{n=1}^{\infty}\frac{k_n\left(\omega^2\tilde{a}_{1n}-k_n\tilde{d}_{1n}\right)}{\Delta_n}\right)\cdot\left(\sum_{n=1}^{\infty}\frac{k_n\left(\omega^2\tilde{a}_{2n}-k_n\tilde{d}_{2n}\right)}{\Delta_n}\right)-$$
$$-\omega^4\left(\sum_{n=1}^{\infty}\frac{k_nb_n}{\Delta_n}\right)^2=0, \qquad (35)$$

Here:

$$\Delta_n=\left(\tilde{a}_{1n}\tilde{a}_{2n}-b_n^2\right)\omega^4-\left(\tilde{a}_{1n}\tilde{d}_{2n}+\tilde{a}_{2n}\tilde{d}_{1n}\right)k_n\omega^2+k_n^2\tilde{d}_{1n}\tilde{d}_{2n},$$

$$\tilde{a}_{in}=a_{in}+k_nk_{0i},\quad \tilde{d}_{in}=\left(D_ik_n^2+T_i\right)k_n^2+g\Delta\rho_i.$$

When the upper plate becomes absolutely solid ($T_1=\infty$), equation (35) will take the form:

$$\sum_{n=1}^{\infty}\frac{k_n}{\tilde{a}_{1n}\omega^2-k_n\tilde{d}_{2n}}=0. \qquad (36)$$

Thus, the frequency equation of joint normal vibrations of elastic rectangular plates and an ideal fluid (30) can be simplified and written in a single form for symmetric and asymmetric frequencies (35). When the upper plate is reborn into a solid one, the equation takes the form of (36).

5. 3. Determining stability conditions of joint vibrations of elastic clamped rectangular plates and an ideal fluid

In works [3, 4] it was shown that to find critical values of mechanical parameters at which loss of stability occurs in the corresponding frequency equation it is sufficient to assume $\omega^2=0$. At $\omega^2=0$ equation (35) breaks down into two equations:

$$\sum_{n=1}^{\infty} 1/\tilde{d}_{1n}=0, \qquad (37)$$

$$\sum_{n=1}^{\infty} 1/\tilde{d}_{2n}=0, \qquad (38)$$

and equation (36) coincides with equation (38).

Since $\tilde{d}_{1n}>0$, equation (37) has no solutions. Equation (38) at $\Delta\rho=\rho_2-\rho_1\geq 0$ will also have no solutions. Accordingly, instability can occur only in the case when $\Delta\rho<0$.

Thus, the stability conditions of equations (35) coincide with the stability conditions of equation (36).

In the case of a membrane ($D_2=0$) and $\Delta\rho<0$, equation (38) in dimensionless form takes the form:

$$\sum_{n=1}^{\infty} 1/\left(n^2-\alpha^2\right)=0, \qquad (39)$$

where $\alpha^2=-4g\Delta\rho a^2/T_2\pi^2>0$.

The number series $\sum_{n=1}^{\infty} 1/\left(n^2-\alpha^2\right)$ for even and odd values of n should be written as follows:

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2-\alpha^2}=\frac{\pi}{4\alpha}\tan\frac{\pi\alpha}{2}, \qquad (40)$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2-\alpha^2}=-\frac{1}{4\alpha^2}\left(\pi\alpha\cot\frac{\pi\alpha}{2}-2\right). \qquad (41)$$

Taking into account (40) ($n=2m-1$), the solution to equation (38) takes the form $a=2l$, and the critical value of tension $T_2=g(\rho_1-\rho_2)a^2/\pi^2l^2$, which at $l=1$ gives the following stability condition:

$$T_2>g\left(\rho_1-\rho_2\right)a^2/\pi^2=0.1013g\left(\rho_1-\rho_2\right)a^2. \qquad (42)$$

Taking into account (41) ($n=2m$), the first root of equation (38) takes the form $\pi a/2=4.493409458$, from which the stability condition follows:

$$T_2>0.0495277g\left(\rho_1-\rho_2\right)a^2. \qquad (43)$$

When $D_2\neq 0$, equation (38) can be rewritten as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^4+\beta^2 n^2-\alpha^4}=0. \qquad (44)$$

Here:

$$\beta^2=4T_2a^2/D_2\pi^2\geq 0,\ \alpha^4=-16g\Delta\rho a^4/D_2\pi^4>0.$$

At $T_2=0$ ($\beta=0$) the numerical series $\sum_{n=1}^{\infty} \frac{1}{n^4+\beta^2n^2-\alpha^4}$ in equation (44) for odd and even values n have the following representation:

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^4-\alpha^4}=\frac{\pi}{8}\frac{\tan(\pi\alpha/2)-\tanh(\pi\alpha/2)}{\alpha^3}, \qquad (45)$$

$$\sum_{k=1}^{\infty} \frac{1}{(2m)^4-\alpha^4}= -\frac{1}{8}\frac{\pi\alpha\cot(\pi\alpha/2)+\pi\alpha\coth(\pi\alpha/2)-4}{\alpha^4}. \qquad (46)$$

Taking into account (45) ($n=2m-1$), the first root of equation (38) takes the form $\pi a/2=3.926602312$, from which the following stability condition follows:

$$D_2>0.0042066g\left(\rho_1-\rho_2\right)a^4. \qquad (47)$$

Taking into account (46) ($n=2m$), the first root of equation (38) takes the form $\pi a/2=5.2676575303$, from which the stability condition follows:

$$D_2>0.00129876g\left(\rho_1-\rho_2\right)a^4. \qquad (48)$$

Let $\Delta\rho=0$ ($\rho_1=\rho_2$). In this case, equation (38) will take the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^4-\beta^2n^2}=0. \qquad (49)$$

Here $\beta^2=4\tilde{T}a^2/D_2\pi^2>0,\ \tilde{T}=-T_2>0$.

The number series $\sum_{n=1}^{\infty} 1/\left[n^2\left(n^2-\beta^2\right)\right]$ in equation (49) for odd and even values n have the representation:

$$\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2\left[(2m-1)^2-\beta^2\right]}=\pi^4\frac{\tan x-x}{32x^3}, \qquad (50)$$

$$\sum_{m=1}^{\infty} \frac{1}{(2m)^2\left[(2m)^2-\beta^2\right]}=-\pi^4\frac{x^2+3x\cot x-3}{96x^4}, \qquad (51)$$

where $x=\pi\beta/2$.

Taking into account (50) ($n=2m-1$), the first root of equation (49) takes the form $x=4.49341$, from which the following stability condition follows:

$$D_2>0.04953\tilde{T}a^2. \qquad (52)$$

Taking into account (51) ($n=2m$), the first root of equation (49) takes the form $x=5.76346$, from which the stability condition follows:

$$D_2>0.03011\tilde{T}a^2. \qquad (53)$$

Thus, the stability conditions at $\rho_1\neq\rho_2$ for the membrane and the plate take the form (42), (43), and (47), (48), respectively, and for $\rho_1=\rho_2$ the stability conditions for the plate take the form of (52), (53).

To verify the established stability conditions, we shall perform numerical calculations of equation (36) for a membrane (D_2=0), where $\Omega^2=\omega^2 b/g$, $\tilde{T}=T/g\rho_2 b^2$, $\tilde{k}_0=k_{02}/\rho_2 b$, $\rho_1\rho_{12}=\rho_1/\rho_2$, $H_i=h_i/b$, $b=2a$.

Fig. 2, 3 show the plots of dependence of the square of the first dimensionless asymmetric frequency on the dimensionless density value ρ_{12} at $\tilde{T}=0.1$ – Fig. 2, and at $\tilde{T}=1.0$ – Fig. 3 for $n=2m-1, m=1,20$. In all figures $H_1=H_2=1$, and the values $\tilde{k}_0=0$ correspond to the upper plot, $\tilde{k}_0=0.5$ – the middle plot and $\tilde{k}_0=1.0$ – the lower plot. It should be noted that the plots for symmetric and asymmetric frequencies do not differ qualitatively and, as a rule, the values of even frequencies are approximately five times greater than the corresponding values for odd frequencies.

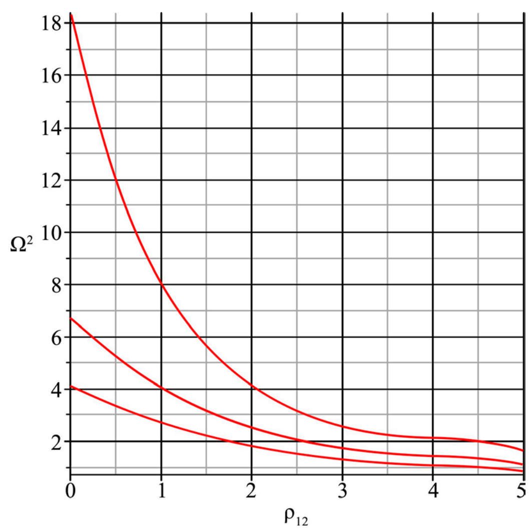

Fig. 2. Dependence of the square of the first dimensionless frequency on ρ_{12} at $\tilde{T}=0.1$

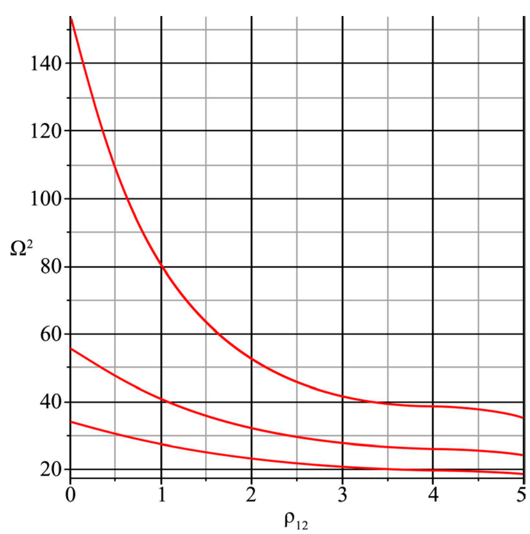

Fig. 3. Dependence of the square of the first dimensionless frequency on ρ_{12} at $\tilde{T}=1$

The above plots of dependence of the square of the first dimensionless frequency on ρ_{12} (Fig. 2, 3) confirm our analytical studies, from which it followed that the oscillation of the membrane and the liquid may be unstable when the heavier liquid is above the lighter liquid (ρ_{12}>1).

6. Discussion of results based on investigating the stability of vibrations of rectangular plates with an ideal fluid

Traditionally, for the study of complex transcendental characteristic equations, a finite number of series terms remain, and the stability conditions are derived from the conditions of positivity of the roots of the frequency equation [1]. According to this method, increasing the series terms makes it impossible to conduct further analytical studies of the conditions of positivity of the roots due to the significant complexity of the resulting characteristic equations. In contrast to conventional approaches, the method proposed in [3, 4] has been further refined in our work. According to it, to find the critical values of mechanical parameters at which stability is lost, it is sufficient to assume $\omega^2=0$ in the corresponding frequency equation. Based on the derived frequency equations (30), (33), (35), (36) and the proposed method, analytical studies of infinite series in equation (38) were carried out and the stability conditions of the combined vibrations of the plates and fluid were established. It is shown that the stability of symmetric and asymmetric vibrations for the membrane and the plate take the form of (42), (43), (47), and (48), respectively, and at $\rho_1=\rho_2$ the stability conditions for the plate take the form of (52), (53). It should be noted that these inequalities do not depend on the elastic properties of the upper membrane or plate, the mass characteristics of the membranes and plates and the depths of filling of liquids. The given approach in the linear statement partially resolves the more complex problem of the stability of nonlinear combined vibrations of the layer and the liquid. Our analytical studies on the problem of plane vibrations give a qualitative assessment of the stability of combined vibrations of plates in an ideal liquid in the case of spatial vibrations.

To verify the established stability conditions, numerical calculations of equation (36) were carried out, which confirmed the analytical studies on stability conditions. The proposed approach makes it possible to derive accurate values of mechanical parameters at which instability will occur, in contrast to conventional approaches, since an increase in the number of series members will lead to the impossibility of specifying the stability conditions.

The limitations of the proposed method include the linear statement of the problem and the assumption of the continuity of plate and fluid oscillations.

Our study is also limited to the consideration of plane oscillations, which gives a limited opportunity to use them in spatial oscillations of plates and fluid.

Further research prospects involve spatially compatible oscillations of plates and an ideal fluid.

7. Conclusions

1. We have constructed and investigated frequency equations of free oscillations of two isotropic elastic rectangular plates in an infinitely long rectangular parallelepiped with an incompressible ideal fluid. The first plate is the upper base of the rectangular parallelepiped, and the second one separates the ideal incompressible fluids having different densities. The contours of the plates may be arbitrarily fixed. The combined oscillations of the elastic rectangular plates and the incompressible fluid are reduced to a system of integrated-dif-

ferential equations, and, accordingly, the forms of deflection of the rectangular plates are represented as the sum of the fundamental solutions to the homogeneous equation for each plate and a partial solution to the inhomogeneous equation.

2. Two separate cases have been investigated by analytical methods: when the contours of the plates are clamped, and when the plates are transformed into membranes. It is shown that this problem has an analytical solution and a discrete spectrum of eigenvalues, and the corresponding eigenfunctions, as a rule, form a complete and orthogonal system of functions.

3. Based on analytical studies of infinite series in the transcendental frequency equation, stability conditions for combined oscillations of rectangular plates and an ideal incompressible fluid have been established. It was shown that instability of oscillations of plates and fluid occurs when a heavier fluid is above a less heavy fluid. The resulting stability conditions for symmetric and asymmetric oscillations of elastic membranes, ideal fluid, and plates do not depend on the elastic parameters of the upper plate, the mass characteristics of the plates, and the depths of filling of liquids. To verify the established stability conditions, numerical calculations of the frequency equation were carried out for the case of a membrane and plots of dependence of the square of the first dimensionless asymmetric frequency on the dimensionless density value were constructed, which confirmed the analytical studies on stability conditions. To verify the resulting stability conditions, numerical calculations of the frequency equation were carried out for the case of a membrane and plots of dependence of the square of the first dimensionless asymmetric frequency on the dimensionless density value were built. From the above plots it follows that with increasing density of the upper liquid the frequencies decrease and may approach zero, which will lead to instability of oscillations. Numerical calculations has confirmed the analytical studies on stability conditions.

Conflicts of interest

The authors declare that they have no conflicts of interest in relation to the current study, including financial, personal, authorship, or any other, that could affect the study, as well as the results reported in this paper.

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Data availability

All data are available, either in numerical or graphical form, in the main text of the manuscript.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

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